

Mirrorfolds with $K3$ fibrations

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ABSTRACT: We study a class of non-geometric string vacua realized as completely soluble superconformal field theory (SCFT). These models are defined as ‘interpolating orbifolds’ of $K3 \times S^1$ by the mirror transformation acting on the $K3$ fiber combined with the half-shift on the S^1 -base. They are variants of the T-folds, the interpolating orbifolds by T-duality transformations, and thus may be called ‘mirrorfolds’. Starting with arbitrary (compact or non-compact) Gepner models for the $K3$ fiber, we construct modular invariant partition functions of general mirrorfold models. In the case of compact $K3$ fiber the mirrorfolds only yield non-supersymmetric string vacua. They exhibit IR instability due to winding tachyon condensation which is similar to the Scherk-Schwarz type circle compactification. When the fiber SCFT is non-compact (say, the ALE space in the simplest case), on the other hand, both supersymmetric and non-supersymmetric vacua can be constructed. The non-compact non-supersymmetric mirrorfolds can get stabilised at the level of string perturbation theory. We also find that in the non-compact supersymmetric mirrorfolds D-branes are *always* non-BPS. These D-branes can get stabilized against both open- and closed-string marginal deformations.

KEYWORDS: Conformal Field Models in String Theory, String Duality, Superstring Vacua.

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1. Introduction and summary

String theory on non-geometric backgrounds has recently been receiving much attention. A particularly accessible class of non-geometric backgrounds is those formulated as fibrations over a base manifold in which the transition functions are built from discrete duality transformations of string theory besides diffeomorphisms. In such models the moduli space of the fibre, when going around non-trivial cycles on the base manifold, picks up monodromies in general; for this reason these string vacua are often called ‘monodrofolds.’ In particular, monodrofolds constructed from T-duality transformations are called ‘T-folds’ [1]. Known examples of T-folds include those arising from flux-compactified type II strings combined with T-duality. These are non-geometric in the sense that while they are locally equipped with geometric structures, globally they are not. It is now increasingly recognised that such backgrounds constitute a natural and essential part of string vacua. For recent topics and developments of non-geometric backgrounds in string theory, see e.g. [2] and references therein.

In studying non-geometric backgrounds that do not necessarily allow intuitive geometric picture, approach by world-sheet conformal field theory (CFT) proves to be extremely powerful. From CFT one may extract various essential information. Firstly, consistency of the string vacua can be examined through modular invariance, locality of vertex operators, etc. One may also find spectra of physical excitations, presence/absence of space-time supersymmetry (SUSY), as well as stability of the system. While limited to the lowest order in the string coupling expansion, CFT gives all-order results in the α' -correction beyond the supergravity approximation. By now, several models of T-folds have been analysed using CFT [3–5]. Detailed study of D-branes in simple T-fold models was also carried out by the present authors in [6], where consistent D-branes on these backgrounds are explicitly constructed in boundary CFT, supporting and supplementing previous observations of [7].

In CFT, T-folds are typically realised as asymmetric interpolating orbifolds. They provide interesting models of string vacua as they generally involve less moduli. Moreover, construction of such CFT models is delicate in general (e.g. achieving modular invariance), giving rise to stringent consistency checks. One may also hope for breaking SUSY while keeping attractive features of SUSY intact in such models, as discussed in [8] based on toroidal models. In the present article we apply techniques of interpolating orbifold CFT to more non-trivial backgrounds of superstring theory. The models we shall study are $K3$ fibrations over an S^1 base with the mirror twist, which we call ‘mirrorfolds,’ following the precedent examples of the monodrofold and T-folds. These are modelled in CFT as interpolating orbifolds of $K3 \times S^1$ with the mirror involution acting on the $K3$ fiber, which may be seen as extensions of the simplest T-folds mentioned above. Note that such orbifolds are possible since $K3$ is self-dual for the mirror symmetry. We shall see that the CFT machinery works well for these non-trivial *curved* fiber spaces. Similar models of string theory compactification involving $K3$ twists are investigated also in [9].

Main outcomes of this paper are summarized as follows:

- We start by considering an arbitrary Gepner model [10] to describe the $K3$ fiber. Besides the standard Gepner models for compact spaces we also treat non-compact models in which gravity decouples [11–13].¹ We elaborate on the construction of modular invariant partition functions in full generality. Careful fixing of phase ambiguity that appears in the mirror involution turns out to be crucial for the modular invariance.
- In the case of the compact $K3$ fiber, the mirrorfolds yield only non-SUSY string vacua. They exhibit IR instability caused by winding tachyon condensation which is similar to the Scherk-Schwarz type circle compactification [15].
- In the case of the non-compact $K3$ fiber (e.g. the ALE spaces), both SUSY and non-SUSY vacua can be constructed. The non-compact, non-SUSY mirrorfolds can be stabilised at the level of perturbative string. Namely, they get stable against arbitrary marginal deformations of normalizable modes.

¹A recent study of the non-compact Gepner-like models has been given in [14].

- The vacua of non-compact SUSY mirrorfolds are stable, of course. However, once putting an arbitrary consistent D-brane on these backgrounds, the space-time SUSY is inevitably broken. We examine the stability of such non-SUSY vacua, and find that the vacuum may become free from instability caused by open string tachyons.

This paper is organized as follows. In section 2, starting with a brief review on the Gepner construction of $K3$, we discuss the construction of modular invariant partition functions describing string theory on the mirrorfolds with compact $K3$ fibrations. In section 3, we study the models with non-compact fibrations. There are several common features in the compact and non-compact mirrorfolds, but there are also remarkable differences. In section 4, we present discussions and outlook for future work. In the appendices we summarize our notations of modular functions and various character formulas appearing in the main text.

We use the convention of $\alpha' = 1$ throughout this paper.

2. Mirrorfolds with $K3$ fibrations

The superconformal system which we focus on in this paper is the *interpolating orbifolds* of the type

$$\frac{K3 \times S^1_{2R}}{\sigma_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}}, \tag{2.1}$$

where σ_{mirror} denotes the mirror involution acting on the $K3$ ‘fiber’, and $\mathcal{T}_{2\pi R}$ denotes the half-shift (let the radius of S^1 be $2R$) along the ‘base’ S^1 -direction:

$$\mathcal{T}_{2\pi R} : Y \longmapsto Y + 2\pi R. \tag{2.2}$$

This conformal system is expected to describe the $K3$ -fibration over the S^1 -base of radius R (reduced by the half-shift $\mathcal{T}_{2\pi R}$), twisted by the mirror-transformation on $K3$. We assume an arbitrary Gepner model for the $K3$ fiber. We will also work with non-compact models in which gravity decouples, in the next section.

2.1 Preliminary: Gepner models for $K3$

In order to establish notations we start with a brief review of the Gepner construction of $K3$:

$$[M_{k_1} \otimes \cdots \otimes M_{k_r}] |_{\mathbb{Z}_N\text{-orbifold}}, \quad \sum_{i=1}^r \frac{k_i}{k_i + 2} = 2, \tag{2.3}$$

where M_k denotes the $\mathcal{N} = 2$ minimal model of level k ($\hat{c} \equiv \frac{c}{3} = \frac{k}{k+2}$), and we set

$$N \equiv \text{L.C.M.}\{k_i + 2 ; i = 1, \dots, r\}. \tag{2.4}$$

The \mathbb{Z}_N -orbifold means to project the Hilbert space onto the subspace with integer $U(1)_R$ -charge. To keep consistency of the conformal field theory, this projection has to be accompanied by the twisted sectors generated by integral spectral flows [16].

The modular invariant of the model (2.3) generically has the following form (assuming the diagonal modular invariant with respect to the spin structure):

$$Z_{K3}(\tau, \bar{\tau}; z, \bar{z}) = \frac{e^{-4\pi \frac{z\bar{z}}{\tau_2}}}{2N} \sum_{I, \tilde{I}} \sum_{\alpha} N_{I, \tilde{I}} F_I^{(\alpha)}(\tau, z) \overline{F_{\tilde{I}}^{(\alpha)}(\tau, z)}, \quad (2.5)$$

where the sum of α runs over the spin structures, and the angle variables z, \bar{z} couple with the total $U(1)_R$ -charge. The factor $e^{-4\pi z\bar{z}/\tau_2}$ ($\tau_2 \equiv \text{Im}\tau, z_2 \equiv \text{Im}z$) is necessary for preserving the modular invariance for $z \neq 0$; it is related to the chiral anomaly of the total $U(1)_R$ current. The chiral blocks $F_I^{(\alpha)}(\tau, z)$ are explicitly written as ‘integral spectral flow orbits’ [16] as

$$F_I^{(\text{NS})}(\tau, z) = \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} q^{a^2} y^{2a} \prod_{i=1}^r \text{ch}_{\ell_i, m_i}^{(\text{NS}), k_i}(\tau, z + a\tau + b), \quad (2.6)$$

for the NS sector. Here I is the collective index: $I \equiv \{(\ell_1, m_1), \dots, (\ell_r, m_r)\}$, and likewise for \tilde{I} ($0 \leq \ell_i \leq k_i, m_i \in \mathbb{Z}_{2(k_i+2)}, \ell_i + m_i \in 2\mathbb{Z}$). The characters $\text{ch}_{\ell_i, m_i}^{(\text{NS}), k_i}(\tau, z)$ of the $\mathcal{N} = 2$ minimal models M_{k_i} are presented in appendix A. The orbits of the other spin structures are defined with $1/2$ -spectral flows:²

$$\begin{aligned} F_I^{(\widetilde{\text{NS}})}(\tau, z) &= F_I^{(\text{NS})}\left(\tau, z + \frac{1}{2}\right), \\ F_I^{(\text{R})}(\tau, z) &= q^{\frac{1}{4}} y F_I^{(\text{NS})}\left(\tau, z + \frac{\tau}{2}\right), \\ F_I^{(\widetilde{\text{R}})}(\tau, z) &= q^{\frac{1}{4}} y F_I^{(\text{NS})}\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right), \end{aligned} \quad (2.7)$$

The multiplicity $N_{I, \tilde{I}}$ is simply,³

$$N_{I, \tilde{I}} \equiv \prod_{i=1}^r \frac{1}{2} \left(\delta_{\ell_i, \tilde{\ell}_i} \delta_{m_i, \tilde{m}_i}^{\delta(2(k_i+2))} + \delta_{\ell_i, k_i - \tilde{\ell}_i} \delta_{m_i, \tilde{m}_i + k_i + 2}^{\delta(2(k_i+2))} \right). \quad (2.8)$$

The summation over $b \in \mathbb{Z}_N$ in (2.6) projects out states that do not satisfy the $U(1)$ -charge integrality condition,

$$Q(I) \equiv \sum_{i=1}^r \frac{m_i}{k_i + 2} \in \mathbb{Z}, \quad (2.9)$$

²In the convention taken here, we do not include extra phase factors originating from the $U(1)_R$ -charge. Consequently, our chiral blocks of the $\widetilde{\text{NS}}$ and $\widetilde{\text{R}}$ sectors have slightly unnatural q -expansions for some I , such as

$$F_I^{(\widetilde{\text{NS}})}(\tau) = -q^{h_I} + a_1 q^{h_I+1} + a_2 q^{h_I+2} \dots$$

Also, the collective index $I \equiv \{(\ell_1, m_1), \dots, (\ell_r, m_r)\}$ encodes quantum numbers of the NS sector even in $F_I^{(\text{R})}$ and $F_I^{(\widetilde{\text{R}})}$. An advantage of this convention is that the modular S-matrices are common to all spin structures.

³We start with the A-type modular invariant for each minimal model M_{k_i} for simplicity. The second term in (2.8) is due to the ‘field identification’

$$\text{ch}_{k_i - \ell_i, m_i + k_i + 2}^{(\text{NS}), k_i}(\tau, z) = \text{ch}_{\ell_i, m_i}^{(\text{NS}), k_i}(\tau, z).$$

which is necessary for the space-time SUSY. By construction $F_I^{(\text{NS})}$ vanishes unless (2.9) is satisfied. On the other hand the integral spectral flow ($a \in \mathbb{Z}_N$) acts on the collective index I as

$$s : I \equiv \{(\ell_1, m_1), \dots, (\ell_r, m_r)\} \longmapsto s(I) \equiv \{(\ell_1, m_1 - 2), \dots, (\ell_r, m_r - 2)\}, \quad (2.10)$$

so obviously,

$$F_{s^n(I)}^{(\alpha)}(\tau, z) = F_I^{(\alpha)}(\tau, z), \quad (\forall n \in \mathbb{Z}). \quad (2.11)$$

In this sense the summation in (2.5) overcounts the chiral blocks and the factor of $1/N$ has been included to compensate the redundancy. The chiral blocks $F_I^{(\text{NS})}(\tau, z)$ defined this way are often useful, since the modular invariance is manifest.

To close this preliminary section, we briefly illustrate the structure of the Hilbert space in the Gepner construction of $K3$. By the above construction the Hilbert spaces are shown to be

$$\mathcal{H}_{\text{Gepner}}^{(\alpha)} = \bigoplus_{n \in \mathbb{Z}_N} \bigoplus_{\substack{I, \tilde{I} \\ Q(I) \in \mathbb{Z}, Q(\tilde{I}) \in \mathbb{Z}}} \left[N_{I, \tilde{I}} \mathcal{H}_{s^n(I), L}^{(\alpha)} \otimes \mathcal{H}_{\tilde{I}, R}^{(\alpha)} \right], \quad (\alpha = \text{NS}, \text{R}) \quad (2.12)$$

where s^n is the actions of the integral spectral flows and $\mathcal{H}_{I, L}^{(\alpha)}$ ($\mathcal{H}_{\tilde{I}, R}^{(\alpha)}$) denotes the left (right) moving Hilbert spaces corresponding to the chiral blocks $F_I^{(\alpha)}(\tau, z)$ ($\overline{F_{\tilde{I}}^{(\alpha)}}(\tau, z)$), that are tensor products of the M_{k_i} minimal model Hilbert spaces. Note that the left-right symmetric primary states lie in the $n = 0$ sector, but we also have many asymmetric primary states generated by the spectral flows. We will later work with the type II string vacua that include chiral spin structures. In those cases the Hilbert spaces (2.12) need be extended by the $1/2$ -spectral flows acting chirally.

In the present $\hat{c} = 2$ case relevant for $K3$, the $\mathcal{N} = 2$ superconformal symmetry is enhanced to the (small) $\mathcal{N} = 4$ by adding the spectral flow operators, which are identified with the $\text{SU}(2)_1$ currents $J^\pm \equiv J^1 \pm iJ^2$ in the $\mathcal{N} = 4$ superconformal algebra (SCA) [16]. Accordingly, the chiral parts of $\mathcal{H}_{\text{Gepner}}^{(\alpha)}$ are decomposed into irreducible representations of $\mathcal{N} = 4$ SCA at level 1, that are classified as follows [17]:

- **massive representations:** $\mathcal{C}_h^{(\text{NS})}, \mathcal{C}_h^{(\text{R})}$

These are non-degenerate representations whose vacua have conformal weights h . The vacuum of $\mathcal{C}_h^{(\text{NS})}$ belongs to the spin 0 representation of the $\text{SU}(2)_1$ -symmetry. The four-fold degenerate vacua of $\mathcal{C}_h^{(\text{R})}$ generate the representation $2[\text{spin } 0] \oplus [\text{spin } 1/2]$. Unitarity requires $h \geq 0$ for $\mathcal{C}_h^{(\text{NS})}$ and $h \geq \frac{1}{4}$ for $\mathcal{C}_h^{(\text{R})}$. The $1/2$ -spectral flow connects $\mathcal{C}_h^{(\text{NS})}$ with $\mathcal{C}_{h+\frac{1}{4}}^{(\text{R})}$.

- **massless representations:** $\mathcal{D}_\ell^{(\text{NS})}, \mathcal{D}_\ell^{(\text{R})}$ ($\ell = 0, 1/2$)

These are degenerate representations whose vacua have conformal weights $h = \ell$ for the NS representations $\mathcal{D}_\ell^{(\text{NS})}$, and $h = \frac{1}{4}$ for the Ramond representations $\mathcal{D}_\ell^{(\text{R})}$; they

belong to the spin ℓ representation of $SU(2)_1$. To be more specific, $\mathcal{D}_0^{(NS)}$ (‘graviton rep.’ or ‘identity rep.’) corresponds to the unique vacuum with $h = 0$, $J_0^3 = 0$, while $\mathcal{D}_{1/2}^{(NS)}$ (‘massless matter rep.’) is generated over doubly degenerated vacua with $h = 1/2$, $J_0^3 = \pm 1/2$. The Ramond sector $\mathcal{D}_{\frac{1}{2}-\ell}^{(R)}$ is connected with $\mathcal{D}_\ell^{(NS)}$ by the $1/2$ -spectral flow.

The decomposition in terms of the $\mathcal{N} = 4$ SCA will be crucial for our construction of the mirrorfolds. The relevant character formulas are summarized in appendix A.

2.2 The mirror twist

Now let us specify the precise action of the mirror involution operator σ_{mirror} in (2.1).

First of all, σ_{mirror} should act as the $U(1)$ -charge conjugation in the right moving $\mathcal{N} = 2$ SCA:

$$\sigma_{\text{mirror},R}(\equiv \sigma_R^{\mathcal{N}=2}): T_R \rightarrow T_R, \quad J_R \rightarrow -J_R, \quad G_R^\pm \rightarrow G_R^\mp, \quad (2.13)$$

while leaving the left moving $\mathcal{N} = 2$ SCA unchanged. Moreover, as the theory is endowed with the $\mathcal{N} = 4$ SCA at level 1, the above $\sigma_{\text{mirror},R}$ acts on the right-moving $\mathcal{N} = 4$ generators $\{T_R, G_R^a, J_R^i\}$ ($a = 0, 1, 2, 3$ and $i = 1, 2, 3$) as well. With the generators of the (total) $\mathcal{N} = 2$ SCA identified as

$$J_R = 2J_R^3, \quad G_R^\pm = G_R^0 \pm iG_R^3, \quad (2.14)$$

the action of the involution is naturally extended on the $\mathcal{N} = 4$ algebra as

$$\begin{aligned} \sigma_{\text{mirror},R}(\equiv \sigma_{1,R}^{\mathcal{N}=4}): \quad T_R &\rightarrow T_R, \quad J_R^1 \rightarrow J_R^1, \quad J_R^i \rightarrow -J_R^i \quad (i = 2, 3), \\ G_R^a &\rightarrow G_R^a \quad (a = 0, 1), \quad G_R^a \rightarrow -G_R^a \quad (a = 2, 3). \end{aligned} \quad (2.15)$$

Here we have introduced the symbol $\sigma_{1,R}^{\mathcal{N}=4}$ for later convenience, and $\sigma_{2,R}^{\mathcal{N}=4}$, $\sigma_{3,R}^{\mathcal{N}=4}$ are defined in the same way by the cyclic permutations of the indices i and a . Since we are assuming the Gepner construction, the total involution $\sigma_{\text{mirror},R}$ is most naturally realised by taking the tensor product of $\mathcal{N} = 2$ involutions in each $\mathcal{N} = 2$ minimal model M_{k_i} ($i = 1, \dots, r$):

$$\sigma_{\text{mirror},R} \equiv \prod_{i=1}^r \sigma_R^{\mathcal{N}=2,(i)}, \quad (2.16)$$

where $\sigma_R^{\mathcal{N}=2,(i)}$ acts on the $\mathcal{N} = 2$ SCA of M_{k_i} as

$$\sigma_R^{\mathcal{N}=2,(i)}: T_R^{(i)} \rightarrow T_R^{(i)}, \quad J_R^{(i)} \rightarrow -J_R^{(i)}, \quad G_R^{\pm,(i)} \rightarrow G_R^{\mp,(i)}. \quad (2.17)$$

It is easy to see that $\sigma_{\text{mirror},R}$ defined in this way acts on the $\mathcal{N} = 4$ SCA as the operator $\sigma_{1,R}^{\mathcal{N}=4}$ above. We shall assume the right-moving operation of the form (2.16) from now on.

The operation of the left-mover $\sigma_{\text{mirror},L}$ still needs to be determined. The simplest guess would be $\sigma_{\text{mirror},L} \equiv \mathbf{1}$, but this does not work. In fact, it turns out that $\sigma_{\text{mirror}}^{\text{naive}} \equiv$

$\mathbf{1} \otimes \sigma_{\text{mirror},R}$ does not leave invariant the closed string Hilbert space of the Gepner model $\mathcal{H}_{\text{Gepner}}$.⁴ We propose that the operator $\sigma_{\text{mirror},L}$ should satisfy following requirements:

1. $\sigma_{\text{mirror}} \equiv \sigma_{\text{mirror},L} \otimes \sigma_{\text{mirror},R}$ acts over $\mathcal{H}_{\text{Gepner}}$ as an involution,

$$\sigma_{\text{mirror}}(\mathcal{H}_{\text{Gepner}}) = \mathcal{H}_{\text{Gepner}}, \quad (\sigma_{\text{mirror}})^2 = \mathbf{1}. \quad (2.18)$$

2. $\sigma_{\text{mirror},L}$ preserves the total $\mathcal{N} = 2$ SCA $\{T_L, J_L, G_L^\pm\}$.
3. The orbifolding by $\sigma_{\text{mirror}} \equiv \sigma_{\text{mirror},L} \otimes \sigma_{\text{mirror},R}$ is compatible with modular invariance.

Due to the second requirement, $\sigma_{\text{mirror},L}$ can only act as a linear transformation on the primary states of the total $\mathcal{N} = 2$ SCA. Especially, it can be regarded as phase changes on a suitably chosen basis of primary states.

We consider following two candidates for $\sigma_{\text{mirror},L}$:

- (i) $\sigma_{\text{mirror},L}$ acting on the $\mathcal{N} = 4$ SCA as the automorphism $\sigma_{3,L}^{\mathcal{N}=4}$. For the $\mathcal{N} = 4$ primary states $|v\rangle_L$, the action of $\sigma_{\text{mirror},L}$ is defined as

$$\sigma_{\text{mirror},L}|v\rangle_L \equiv \begin{cases} \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = 0), \\ J_{L,0}^+ \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = |v\rangle_L), \\ -J_{L,0}^- \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = -|v\rangle_L), \end{cases} \quad (2.19)$$

where $J_L^\pm \equiv J_L^1 \pm iJ_L^2$ are the SU(2) currents in the $\mathcal{N} = 4$ SCA.

- (ii) $\sigma_{\text{mirror},L}$ preserving the $\mathcal{N} = 4$ SCA. For the $\mathcal{N} = 4$ primary states $|v\rangle_L$, the action of $\sigma_{\text{mirror},L}$ is defined as

$$\sigma_{\text{mirror},L}|v\rangle_L \equiv \begin{cases} \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = 0), \\ J_{L,0}^+ \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = |v\rangle_L), \\ J_{L,0}^- \prod_{i=1}^r \sigma_L^{\mathcal{N}=2,(i)}|v\rangle_L, & (2J_{L,0}^3|v\rangle_L = -|v\rangle_L). \end{cases} \quad (2.20)$$

It is easy to verify that these two candidates indeed satisfy the first and second conditions given above. Checking the modular invariance is a non-trivial task and we will discuss it from now on.

⁴For example, pick up a symmetric primary state of the form

$$|v\rangle \equiv \prod_i |\ell_i, m_i, s_i\rangle_L \otimes \prod_i |\ell_i, m_i, s_i\rangle_R, \quad (\ell_i + m_i + s_i \in 2\mathbb{Z}, \quad m_i \in \mathbb{Z}_{2(k_i+2)}, \quad s_i \in \mathbb{Z}_4).$$

The above $\sigma_{\text{mirror}}^{\text{naive}}$ acts on it as

$$\sigma_{\text{mirror}}^{\text{naive}}|v\rangle = \prod_i |\ell_i, m_i, s_i\rangle_L \otimes \prod_i |\ell_i, -m_i, -s_i\rangle_R,$$

which in general is not a state in $\mathcal{H}_{\text{Gepner}}$.

2.3 Modular invariant partition functions of the mirrorfolds with compact $K3$ fibers

We shall construct modular invariant partition functions of the mirrorfolds (2.1). We take an arbitrary Gepner model describing a compact $K3$ fiber. We assume (2.16) for $\sigma_{\text{mirror},R}$, and adopt the first candidate (2.19) for $\sigma_{\text{mirror},L}$.

Before discussing the construction of the modular invariant, we need to find the $\mathcal{N} = 4$ character formulas twisted by $\sigma_{i,L}^{\mathcal{N}=4}$ ($\sigma_{i,R}^{\mathcal{N}=4}$). We first focus on the $\sigma_{3,L}^{\mathcal{N}=4}$ -twist. We express the spatial and temporal boundary conditions as $[S, T]$, $S, T \in \mathbb{Z}_2$ ($S, T = 0$ means no twist, while $S, T = 1$ indicates twisting by $\sigma_{3,L}^{\mathcal{N}=4}$). The desired character formulas are readily obtained by starting with the temporal twist boundary condition $[S, T] = [0, 1]$ (i.e. inserting $\sigma_{3,L}^{\mathcal{N}=4}$ into the trace), which results in an extra phase factor $(-1)^n$ in the n -th spectral flow sector. For $[S, T] = [0, 1]$ the formula (A.13) is thus replaced by

$$\begin{aligned} \text{ch}_{*,[0,1]}^{\mathcal{N}=4,(\text{NS})}(*; \tau, z) &\equiv \text{Tr}_{\mathcal{H}}[\sigma_{3,L}^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} y^{2J_0^3}] \\ &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} y^{2n} \text{ch}_*^{\mathcal{N}=2,(\text{NS})}(*; \tau, z + n\tau). \end{aligned} \quad (2.21)$$

Here \mathcal{H} denotes the representation space of $\mathcal{C}_h^{(\text{NS})}$, $\mathcal{D}_0^{(\text{NS})}$ or $\mathcal{D}_{1/2}^{(\text{NS})}$. We spell out explicit results in each case:

massive representation:

$$\text{ch}_{[0,1]}^{\mathcal{N}=4,(\text{NS})}(h; \tau, z) = q^{h - \frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} y^n \frac{\theta_3(\tau, z)}{\eta(\tau)^3} = q^{h - \frac{1}{8}} \frac{\theta_3(\tau, z) \theta_4(\tau, z)}{\eta(\tau)^3}, \quad (2.22)$$

massless representations:

$$\text{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{NS})}\left(\ell = \frac{1}{2}; \tau, z\right) = q^{-1/8} \sum_{n \in \mathbb{Z}} (-1)^{n+1} \frac{1}{1 + yq^{n-1/2}} q^{\frac{n^2}{2}} y^n \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \quad (2.23)$$

$$\begin{aligned} \text{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{NS})}(\ell = 0; \tau, z) &= q^{-1/8} \sum_{n \in \mathbb{Z}} (-1)^n \frac{(1 - q)q^{\frac{n^2}{2} + n - \frac{1}{2}} y^{n+1}}{(1 + yq^{n+1/2})(1 + yq^{n-1/2})} \frac{\theta_3(\tau, z)}{\eta(\tau)^3} \\ &\equiv q^{-1/8} \frac{\theta_3(\tau, z) \theta_4(\tau, z)}{\eta(\tau)^3} \equiv \text{ch}_{[0,1]}^{\mathcal{N}=4,(\text{NS})}(h = 0; \tau, z). \end{aligned} \quad (2.24)$$

The second line of (2.24) follows from identity

$$\frac{(1 - q)q^{n - \frac{1}{2}} y}{(1 + yq^{n+1/2})(1 + yq^{n-1/2})} = 1 - \frac{1}{1 + yq^{n - \frac{1}{2}}} - \frac{yq^{n + \frac{1}{2}}}{1 + yq^{n + \frac{1}{2}}}. \quad (2.25)$$

Specializing to $z = 0$, we further obtain

$$\text{ch}_{[0,1]}^{\mathcal{N}=4,(\text{NS})}(h; \tau, 0) = q^{h-1/8} \frac{\theta_3(\tau) \theta_4(\tau)}{\eta(\tau)^3} \equiv q^{h-1/8} \frac{2}{\theta_2(\tau)} \equiv \chi_{[0,1]}(p; \tau), \quad \left(h = \frac{p^2}{2} + \frac{1}{8}\right), \quad (2.26)$$

$$\text{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{NS})}(\ell = 1/2; \tau, 0) = q^{-1/8} \sum_{n \in \mathbb{Z}} (-1)^{n+1} \frac{1}{1 + q^{n-1/2}} q^{\frac{n^2}{2}} \frac{\theta_3(\tau)}{\eta(\tau)^3} \equiv 0, \quad (h = 1/2), \quad (2.27)$$

$$\text{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{NS})}(\ell=0; \tau, 0) = q^{-1/8} \frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^3} \equiv q^{-1/8} \frac{2}{\theta_2(\tau)} \equiv \chi_{[0,1]}(p=i/2; \tau), (h=0), \quad (2.28)$$

where we used the abbreviation $\theta_i(\tau) \equiv \theta_i(\tau, 0)$, and $\chi_{[0,1]}(p; \tau)$ is the $\mathcal{N} = 2$ twisted character of $\hat{c} = 2$ (B.4).

The character formulas for the other boundary conditions are derived by acting modular transformations successively, at least for $z = 0$. We denote the spin structures and the boundary conditions of $\sigma_{3,L}^{\mathcal{N}=4}$ such as $\{\text{NS}, [S, T]\}$. Starting with the character formula of $\{\text{NS}, [0, 1]\}$ given above, it turns out that there are three types of non-trivial characters $\chi_{[0,1]}(p; \tau)$, $\chi_{[1,0]}(p; \tau)$, $\chi_{[1,1]}(p; \tau)$ (see (B.4)):

$$\begin{aligned} \{\text{NS}, [0, 1]\}, \{\widetilde{\text{NS}}, [0, 1]\} &: \chi_{[0,1]}(p; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_2(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{1}{8}\right), \\ \{\text{NS}, [1, 0]\}, \{\text{R}, [1, 0]\} &: \chi_{[1,0]}(p; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_4(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{1}{4}\right), \\ \{\widetilde{\text{NS}}, [1, 1]\}, \{\text{R}, [1, 1]\} &: \chi_{[1,1]}(p; \tau) \equiv \frac{2q^{\frac{p^2}{2}}}{\theta_3(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{1}{4}\right). \end{aligned} \quad (2.29)$$

These are the building blocks necessary for our construction of the mirrorfold modular invariants. There still remain boundary conditions that are connected to $\{\text{R}, [0, 1]\}$ and $\{\widetilde{\text{R}}, [0, 1]\}$ by modular transformations. We need some further technicality to obtain such twisted characters, and the complete list of the $\mathcal{N} = 4$ twisted characters are given in appendix D. For our purposes, however, only the ones given in (2.29) are needed.

What about the $\sigma_{1,L}^{\mathcal{N}=4}$ -twisting? Since the $\sigma_{1,L}^{\mathcal{N}=4}$ -twist acts as $J(\equiv 2J^3) \rightarrow -J$ on the $U(1)_R$ -current of the underlying $\mathcal{N} = 2$ SCA, none of the spectrally flowed sectors contribute to the $\sigma_{1,L}^{\mathcal{N}=4}$ -twisted characters. Recalling that the $\mathcal{N} = 4$ SCA is obtained by extending the $\mathcal{N} = 2$ SCA by adding the spectral flow operators, we conclude that the $\sigma_{1,L}^{\mathcal{N}=4}$ -twisted $\mathcal{N} = 4$ characters must coincide with the twisted $\mathcal{N} = 2$ characters of $\hat{c} = 2$ (B.4). This means that we are simply led to the same classification of $\sigma_1^{\mathcal{N}=4}$ -twisted characters as (2.29).

In this sense it seems natural to express the above twisted character $\chi_{[0,1]}(p; \tau)$ in two different ways, one that is natural for the $\sigma_3^{\mathcal{N}=4}$ -twist, and the other for the $\sigma_1^{\mathcal{N}=4}$ twist:

$$\begin{aligned} \chi_{[0,1]}(p; \tau) &= \text{Tr}_{\mathcal{C}_h^{\text{(NS)}}}[\sigma_{1,L}^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}}] = \frac{q^{h-1/8}}{\eta(\tau)} \cdot \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}} \cdot \sqrt{\frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^2}} \\ &\equiv q^{h-\frac{1}{4}} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})(1 - q^{n-1/2})}{\prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)}, \end{aligned} \quad (2.30)$$

$$\begin{aligned} \chi_{[0,1]}(p; \tau) &= \text{Tr}_{\mathcal{C}_h^{\text{(NS)}}}[\sigma_{3,L}^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}}] = q^{h-1/8} \theta_4(\tau) \cdot \frac{\theta_3(\tau)}{\eta(\tau)^3} \\ &\equiv q^{h-\frac{1}{4}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \frac{\prod_{n=1}^{\infty} (1 + q^{n-1/2})^2}{\prod_{n=1}^{\infty} (1 - q^n)^2}. \end{aligned} \quad (2.31)$$

The equality of (2.30) and (2.31) is immediately checked by the Euler identity $\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) = 2\eta(\tau)^3$. The equivalence of the $\sigma_3^{\mathcal{N}=4}$ - and $\sigma_1^{\mathcal{N}=4}$ -twisted character formulas (2.29) is anticipated from the existence of an automorphism interpolating $\sigma_3^{\mathcal{N}=4}$ and

$\sigma_1^{\mathcal{N}=4}$ within the $\mathcal{N} = 4$ SCA. Similar results for the other boundary condition (such as $\{\mathbf{R}, [0, 1]\}$), which are less trivial, are discussed in appendix D.

We now proceed to our main analysis. The chiral blocks for each sector of the $K3$ twisted by $\sigma_{\text{mirror}} \equiv \sigma_{\text{mirror},L} \otimes \sigma_{\text{mirror},R}$ are obtained as follows.

The right-mover. After making the $\sigma_{\text{mirror},R}$ -insertion only the spectral flow orbits of type $\{(\ell_1, 0), \dots, (\ell_r, 0)\}$ belonging to the NS or $\widetilde{\text{NS}}$ sectors survive, while none of the orbits in the R nor $\widetilde{\text{R}}$ sectors contributes.⁵ The resultant chiral blocks are

$$\overline{\chi_{\mathbf{l},[S,T]}^{\mathbf{k}}(\tau)} \equiv \prod_{i=1}^r \overline{\chi_{\ell_i,[S,T]}^{k_i}(\tau)}, \quad (2.32)$$

$$\mathbf{l} \equiv (\ell_1, \dots, \ell_r), \quad \mathbf{k} \equiv (k_1, \dots, k_r),$$

where $\chi_{\ell_i,[S,T]}^{k_i}(\tau)$ are the twisted characters of the $\mathcal{N} = 2$ minimal models (B.6). The chiral blocks (2.32) can also be expressed in terms of the twisted $\mathcal{N} = 4$ characters $\chi_{[S,T]}(p; \tau)$ (2.29). For example, picking up the spectral flow orbit $\mathbf{l} \equiv \{(\ell_1, 0), \dots, (\ell_r, 0)\}$, we may write,

$$\begin{aligned} \overline{\chi_{\mathbf{l},[0,1]}^{\mathbf{k}}(\tau)} &\equiv \text{Tr}_{\text{orbit of } \mathbf{l}} \left[\sigma_{\text{mirror},R} \bar{q}^{\tilde{L}_0 - \frac{1}{4}} \right] \\ &= \sum_{n=0}^{\infty} a_{n,\mathbf{l}} \overline{\chi_{[0,1]}(p_{n,\mathbf{l}}; \tau)} \quad \left(\frac{p_{n,\mathbf{l}}^2}{2} + \frac{1}{8} = h_{\mathbf{l}} + n, \quad h_{\mathbf{l}} \equiv \sum_i \frac{\ell_i(\ell_i + 2)}{4(k_i + 2)} \right) \\ &= \sum_{n=0}^{\infty} a_{n,\mathbf{l}} \bar{q}^{h_{\mathbf{l}} + n - \frac{1}{8}} \frac{2}{\theta_2(\tau)}, \end{aligned} \quad (2.33)$$

with $a_{n,\mathbf{l}} \in \mathbb{Z}$, $a_{0,\mathbf{l}} = 1$. It is convenient to introduce a function $f_{\mathbf{l},[0,1]}^{\mathbf{k}}(\tau)$ defined by power series

$$f_{\mathbf{l},[0,1]}^{\mathbf{k}}(\tau) \equiv \sum_{n=0}^{\infty} a_{n,\mathbf{l}} q^{h_{\mathbf{l}} + n - \frac{1}{8}}, \quad (2.34)$$

or more concisely,

$$\chi_{\mathbf{l},[0,1]}^{\mathbf{k}}(\tau) = \frac{2}{\theta_2(\tau)} f_{\mathbf{l},[0,1]}^{\mathbf{k}}(\tau). \quad (2.35)$$

Similar functions for the other boundary conditions $f_{\mathbf{l},[1,0]}^{\mathbf{k}}$, $f_{\mathbf{l},[1,1]}^{\mathbf{k}}$ are defined in the same way,

$$\chi_{\mathbf{l},[1,0]}^{\mathbf{k}}(\tau) = \frac{2}{\theta_4(\tau)} f_{\mathbf{l},[1,0]}^{\mathbf{k}}(\tau), \quad \chi_{\mathbf{l},[1,1]}^{\mathbf{k}}(\tau) = \frac{2}{\theta_3(\tau)} f_{\mathbf{l},[1,1]}^{\mathbf{k}}(\tau). \quad (2.36)$$

⁵The easiest way to see this is to recall that the $\mathcal{N} = 2$ involution $\sigma^{\mathcal{N}=2}$ acts on primary states of the $\mathcal{N} = 2$ minimal model M_k as

$$\sigma^{\mathcal{N}=2} : |\ell, m, s\rangle \longmapsto |\ell, -m, -s\rangle, \quad (\ell + m + s \in 2\mathbb{Z}, \quad \ell = 0, \dots, k, \quad m \in \mathbb{Z}_{2(k+2)}, \quad s \in \mathbb{Z}_4).$$

In the R-sector we have $s = \pm 1 \pmod{4}$ and thus $\sigma^{\mathcal{N}=2}$ does not have any fixed point. This means that the $\sigma^{\mathcal{N}=2}$ -inserted traces always vanish in the R-sector.

By construction we find,

$$\mathrm{Tr}_{\mathcal{N}=4 \text{ vacua of } \mathbf{1}}[\sigma_{\text{mirror},R} \bar{q}^{\tilde{L}_0 - \frac{1}{4}}] = \overline{f_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau)}, \quad (2.37)$$

where the trace is taken over the $\mathcal{N} = 4$ primary states belonging to the orbit $\mathbf{1}$. Modular properties of functions $f_{\mathbf{1},[S,T]}^{\mathbf{k}}(\tau)$ are immediately read off from those of the $\mathcal{N} = 2$ twisted minimal characters $\chi_{\ell_i,[S,T]}^{k_i}$. See formulas (B.9).

The left-mover. Since we have assumed (2.19) for $\sigma_{\text{mirror},L}$, it is convenient to decompose the chiral blocks into the $\mathcal{N} = 4$ irreducible representations. Contributions from the massless rep. $\mathcal{D}_{1/2}^{(\text{NS})}$ ($Q = \pm 1$) trivially vanish because of (2.27). Also, the Ramond rep. $\mathcal{D}_{1/2}^{(\text{R})}$ does not contribute because

$$\begin{aligned} \mathrm{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{R})}(\ell = 1/2; \tau, 0) &\left(\equiv \mathrm{Tr}_{\mathcal{D}_{1/2}^{(\text{R})}} \left[\sigma_{3,L}^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] \right) \equiv q^{\frac{1}{4}} \mathrm{ch}_{0,[0,1]}^{\mathcal{N}=4,(\text{NS})} \left(\ell = 0; \tau, \frac{\tau}{2} \right) \\ &= q^{-\frac{1}{8}} \frac{i\theta_1(\tau, 0)\theta_2(\tau, 0)}{\eta(\tau)^3} = 0, \end{aligned} \quad (2.38)$$

where we have used (2.24) in the second line. Thus, possible non-vanishing contributions only come from representations generated by neutral $\mathcal{N} = 4$ primary states ($Q = 0$). Since $\sigma_{\text{mirror},L}$ acts as $\prod_i \sigma_L^{\mathcal{N}=2,(i)}$ on neutral $\mathcal{N} = 4$ primaries, again we find only the contributions from spectral flow orbits $\mathbf{1} \equiv \{(\ell_1, 0), \dots, (\ell_r, 0)\}$ in the NS ($\widetilde{\text{NS}}$) sector, and no contribution from the R ($\widetilde{\text{R}}$) sector. We thus obtain,

$$\mathrm{Tr}_{\mathcal{N}=4 \text{ vacua of } \mathbf{1}}[\sigma_{\text{mirror},L} q^{L_0 - \frac{1}{4}}] = f_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau), \quad (2.39)$$

$$\mathrm{Tr}_{\mathcal{N}=4 \text{ vacua of other neutral orbits}}[\sigma_{\text{mirror},L} q^{L_0 - \frac{1}{4}}] = 0. \quad (2.40)$$

As we observed above, the $\sigma_3^{\mathcal{N}=4}$ -twisted characters are equal to the $\sigma_1^{\mathcal{N}=4}$ -twisted ones in the relevant sectors. Therefore, we conclude that the chiral blocks of the left-mover formally take the same form as the right-mover:

$$\mathrm{Tr}_{\text{orbit of } \mathbf{1}}[\sigma_{\text{mirror},L} q^{L_0 - \frac{1}{4}}] = f_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau) \cdot \frac{2}{\theta_2(\tau)} \left(\equiv f_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau) \cdot \frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^3} \right). \quad (2.41)$$

The same happens for other boundary conditions $[1, 0]$, $[1, 1]$ due to modular transformations. This fact makes the modular invariance of the $K3$ mirrorfolds possible.

At this stage, we may describe the modular invariant partition functions for the string vacua of our mirrorfold model (2.1) (\times flat space-time $\mathbb{R}^{4,1}$). It can be written in the form,

$$Z(\tau, \bar{\tau}) = Z^{\text{u}}(\tau, \bar{\tau}) + Z^{\text{t}}(\tau, \bar{\tau}), \quad (2.42)$$

where Z^{u} is the partition function of the untwisted sector, and Z^{t} denotes contributions of the twisted sectors⁶ including both temporal and spatial twists by $\sigma_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}$.

⁶In the literature it is traditional to use this term for sectors with only the spatial twist(s). Here, we define Z^{t} to include also the temporal-twisted sector. This somewhat non-standard usage is for computational convenience and hopefully no confusion arises.

Assuming the type II string vacuum, the partition function for the untwisted sector is given as

$$Z^u(\tau, \bar{\tau}) = \frac{1}{2} \cdot \frac{1}{4N} \sum_{\alpha, \tilde{\alpha}} \sum_{I, \tilde{I}} \epsilon(\alpha) \epsilon_{A \text{ or } B}(\tilde{\alpha}) \left(\frac{\theta_{[\alpha]}}{\eta} \right)^2 \overline{\left(\frac{\theta_{[\tilde{\alpha}]}}{\eta} \right)^2} N_{I, \tilde{I}} F_I^{(\alpha)}(\tau) \overline{F_{\tilde{I}}^{(\tilde{\alpha})}(\tau)} \cdot \frac{1}{\tau_2^{3/2} |\eta|^6} Z_{2R}(\tau, \bar{\tau}), \quad (2.43)$$

where we set $\theta_{[\text{NS}]} = \theta_3$, $\theta_{[\widetilde{\text{NS}}]} = \theta_4$, $\theta_{[\text{R}]} = \theta_2$ ($\theta_{[\widetilde{\text{R}}]} = i\theta_1 \equiv 0$), and $\epsilon(\text{NS}) = \epsilon(\widetilde{\text{R}}) = +1$, $\epsilon(\widetilde{\text{NS}}) = \epsilon(\text{R}) = -1$. For the right-mover, we set $\epsilon_B(\tilde{\alpha}) = \epsilon(\tilde{\alpha})$ for type IIB, while $\epsilon_A(\text{NS}) = +1$, $\epsilon_A(\widetilde{\text{NS}}) = \epsilon_A(\text{R}) = \epsilon_A(\widetilde{\text{R}}) = -1$ for type IIA. We used abbreviation $F_I^{(\alpha)}(\tau) \equiv F_I^{(\alpha)}(\tau, 0)$ here.

Free non-compact bosons in the (transverse part of) $\mathbb{R}^{4,1}$ contribute to the factor $1/\tau_2^{3/2} |\eta|^6$. The familiar partition function of a compact boson with radius R is

$$Z_R(\tau, \bar{\tau}) = \frac{R}{\sqrt{\tau_2} |\eta(\tau)|^2} \sum_{w, m \in \mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2} |w\tau + m|^2}. \quad (2.44)$$

We further introduce

$$Z_{R,(a,b)}(\tau, \bar{\tau}) = \frac{R}{\sqrt{\tau_2} |\eta(\tau)|^2} e^{-\frac{\pi R^2}{\tau_2} |a\tau + b|^2}, \quad (a, b \in \mathbb{Z}), \quad (2.45)$$

which describes the contribution from each winding sector

$$\begin{aligned} Y(z+1, \bar{z}+1) &= Y(z, \bar{z}) + 2\pi a R, \\ Y(z+\tau, \bar{z}+\bar{\tau}) &= Y(z, \bar{z}) + 2\pi b R. \end{aligned} \quad (2.46)$$

The sectors with even windings $a, b \in 2\mathbb{Z}$ are identified with the untwisted sectors, leading to

$$\sum_{a, b \in 2\mathbb{Z}} Z_{R,(a,b)}(\tau, \bar{\tau}) = \frac{1}{2} Z_{2R}(\tau, \bar{\tau}). \quad (2.47)$$

The partition function of the twisted sectors is much more complicated. Requiring modular invariance, the partition function $Z^t(\tau, \bar{\tau})$ is expected to be of the form,

$$Z^t(\tau, \bar{\tau}) = \sum_{\substack{a \in 2\mathbb{Z}+1 \\ \text{or } b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \Xi_{(a,b)}(\tau, \bar{\tau}), \quad (2.48)$$

where $\Xi_{(a,b)}(\tau, \bar{\tau})$ are some functions that behave covariantly under modular transformations,

$$\Xi_{(a,b)}(\tau+1, \bar{\tau}+1) = \Xi_{(a,b+a)}(\tau, \bar{\tau}), \quad \Xi_{(a,b)}(-1/\tau, -1/\bar{\tau}) = \Xi_{(b,-a)}(\tau, \bar{\tau}). \quad (2.49)$$

The winding dependence of $\Xi_{(a,b)}(\tau, \bar{\tau})$ primarily originates from the σ_{mirror} -twisting in the $K3$ -sector:

- (i) $a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1$: the sector with temporal twisting by σ_{mirror} .
- (ii) $a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}$: the sector with spatial twisting by σ_{mirror} .
- (iii) $a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1$: the sector with both temporal and spatial twisting by σ_{mirror} .

The calculation for each chiral block of the $K3$ sector is carried out based on the above argument. After summing over the chiral spin structures, we find following partition functions for the twisted sectors:

$$\begin{aligned}
 Z^t(\tau, \bar{\tau}) &= \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ \text{or } b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^{3/2} |\eta|^6} Z_{(a,b)}^f(\tau, \bar{\tau}) \sum_{\mathbf{1}, \bar{\mathbf{1}}} N_{\mathbf{1}, \bar{\mathbf{1}}}^{[[a],[b]]} \chi_{\mathbf{1}, [[a],[b]]}^{\mathbf{k}}(\tau) \overline{\chi_{\mathbf{1}, [[a],[b]]}^{\mathbf{k}}(\tau)} \\
 &\equiv \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z} \\ b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^{3/2} |\eta|^6} \left| \left(\frac{\theta_3}{\eta} \right)^2 - (-1)^{\frac{a}{2}} \left(\frac{\theta_4}{\eta} \right)^2 \right|^2 \\
 &\quad \times \sum_{\mathbf{1}, \bar{\mathbf{1}}} N_{\mathbf{1}, \bar{\mathbf{1}}}^{[0,1]} f_{\mathbf{1}, [0,1]}^{\mathbf{k}}(\tau) \theta_4(\tau) \frac{\theta_3(\tau)}{\eta(\tau)^3} \cdot f_{\mathbf{1}, [0,1]}^{\mathbf{k}}(\tau) \frac{1}{\eta(\tau)} \sqrt{\frac{2\eta(\tau)}{\theta_2(\tau)}} \sqrt{\frac{\theta_3(\tau)\theta_4(\tau)}{\eta(\tau)^2}} \\
 &+ \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ b \in 2\mathbb{Z}}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^{3/2} |\eta|^6} \left| \left(\frac{\theta_3}{\eta} \right)^2 - (-1)^{\frac{b}{2}} \left(\frac{\theta_2}{\eta} \right)^2 \right|^2 \\
 &\quad \times \sum_{\mathbf{1}, \bar{\mathbf{1}}} N_{\mathbf{1}, \bar{\mathbf{1}}}^{[1,0]} f_{\mathbf{1}, [1,0]}^{\mathbf{k}}(\tau) \theta_2(\tau) \frac{\theta_3(\tau)}{\eta(\tau)^3} \cdot f_{\mathbf{1}, [1,0]}^{\mathbf{k}}(\tau) \frac{1}{\eta(\tau)} \sqrt{\frac{2\eta(\tau)}{\theta_4(\tau)}} \sqrt{\frac{\theta_2(\tau)\theta_3(\tau)}{\eta(\tau)^2}} \\
 &+ \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^{3/2} |\eta|^3} \left| \left(\frac{\theta_4}{\eta} \right)^2 + i(-1)^{\frac{a+b}{2}} \left(\frac{\theta_2}{\eta} \right)^2 \right|^2 \\
 &\quad \times \sum_{\mathbf{1}, \bar{\mathbf{1}}} N_{\mathbf{1}, \bar{\mathbf{1}}}^{[1,1]} f_{\mathbf{1}, [1,1]}^{\mathbf{k}}(\tau) \theta_2(\tau) \frac{\theta_4(\tau)}{\eta(\tau)^3} \cdot f_{\mathbf{1}, [1,1]}^{\mathbf{k}}(\tau) \frac{1}{\eta(\tau)} \sqrt{\frac{2\eta(\tau)}{\theta_3(\tau)}} \sqrt{\frac{\theta_4(\tau)\theta_2(\tau)}{\eta(\tau)^2}},
 \end{aligned} \tag{2.50}$$

where we set $[a] \in \mathbb{Z}_2$, $a \equiv [a] \pmod{2}$, and $N_{\mathbf{1}, \bar{\mathbf{1}}}^{[S,T]}$ are suitably chosen coefficients, which will be specified below. In the second line we emphasized the $\mathcal{N} = 4$ structure in the $K3$ -sector. We have adopted an apparent asymmetric form as in [4], which seems natural if we recall $\sigma_{\text{mirror}, R} \sim \sigma_{1,R}^{\mathcal{N}=4}$, $\sigma_{\text{mirror}, L} \sim \sigma_{3,L}^{\mathcal{N}=4}$ when acting on the $\mathcal{N} = 4$ SCA.

Let us further elaborate contributions from each sector.

- (1) **S^1 -sector (bosonic):** The bosonic part of the S^1 -direction is represented by the functions $Z_{R,(a,b)}(\tau, \bar{\tau})$, ($a, b \in \mathbb{Z}$). Sectors with $a \in 2\mathbb{Z} + 1$ or $b \in 2\mathbb{Z} + 1$ correspond to twisted sectors, while contributions from $a, b \in 2\mathbb{Z}$ are included in the partition function of the untwisted sector $Z^u(\tau, \bar{\tau})$.

	[0,1]	[1,0]	[1,1]
NS	$\chi_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau)$	$\chi_{\mathbf{1},[1,0]}^{\mathbf{k}}(\tau)$	0
$\widetilde{\text{NS}}$	$\chi_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau)$	0	$\chi_{\mathbf{1},[1,1]}^{\mathbf{k}}(\tau)$
R	0	$\chi_{\mathbf{1},[1,0]}^{\mathbf{k}}(\tau)$	$\chi_{\mathbf{1},[1,1]}^{\mathbf{k}}(\tau)$
$\widetilde{\text{R}}$	0	0	0

Table 1: Relation between the σ_{mirror} -twists and the spin structures. Note that $f_{\mathbf{1},[\alpha,\beta]}^{\mathbf{k}}(\tau)$ are subject to the same relations.

(2) ***K3*-sector:** As discussed above, the chiral blocks are written in the form of

$$\sum_{\mathbf{1},\tilde{\mathbf{1}}} N_{\mathbf{1},\tilde{\mathbf{1}}}^{[[a],[b]]} \chi_{\mathbf{1},[[a],[b]]}^{\mathbf{k}}(\tau) \overline{\chi_{\tilde{\mathbf{1}},[[a],[b]]}^{\mathbf{k}}(\tau)} \equiv \sum_{\mathbf{1},\tilde{\mathbf{1}}} N_{\mathbf{1},\tilde{\mathbf{1}}}^{[[a],[b]]} f_{\mathbf{1},[[a],[b]]}^{\mathbf{k}}(\tau) \overline{f_{\tilde{\mathbf{1}},[[a],[b]]}^{\mathbf{k}}(\tau)} \left| \frac{2}{\theta_{[[a],[b]]}(\tau)} \right|^2, \quad (2.51)$$

$$\theta_{[0,1]} \equiv \theta_2, \quad \theta_{[1,0]} \equiv \theta_4, \quad \theta_{[1,1]} \equiv \theta_3.$$

The relation between the spin structure and the σ_{mirror} -twisting is slightly non-trivial, and is summarized in table 1. As was already illustrated, the chiral block for boundary condition $[0, 1]$ ($a \in 2\mathbb{Z}$, $b \in 2\mathbb{Z} + 1$) includes only the NS and $\widetilde{\text{NS}}$ -sectors, contributing the same character function $\chi_{\mathbf{1},[0,1]}^{\mathbf{k}}(\tau)$. There is no contribution from the R-sector for this boundary condition.

The blocks (2.51) are clearly modular covariant with respect to the indices a, b , but the modular transformations generate non-trivial mixing of the quantum numbers $\mathbf{1}$ and $\tilde{\mathbf{1}}$. We thus have to choose the coefficients $N_{\mathbf{1},\tilde{\mathbf{1}}}^{[[a],[b]]}$ carefully. This is accomplished by requiring (in addition to the modular invariance) that the orbifold projection $\frac{1+\sigma_{\text{mirror}}}{2}$ acts correctly on the total Hilbert space. To this aim it is convenient to classify the *K3* Gepner models into the following two cases.⁷

(i) **At least one of k_i 's is odd.** It is easiest to look at the $[0, 1]$ -sector (σ_{mirror} -insertion). The problem translates into finding out terms that survive the σ_{mirror} -insertion in the trace out of the spectral flow orbits

$$\{(\ell_1, 0), \dots, (\ell_r, 0)\}_L \otimes \{(\tilde{\ell}_1, 2n), \dots, (\tilde{\ell}_r, 2n)\}_R \quad (n \in \mathbb{Z}_N). \quad (2.52)$$

Under the assumption on k_i , we see that only the terms of the form

$$\prod_i \chi_{\ell_i, [0,1]}^{k_i}(\tau) \overline{\chi_{\tilde{\ell}_i, [0,1]}^{k_i}(\tau)} \quad (2.53)$$

do survive. We thus obtain

$$N_{\mathbf{1},\tilde{\mathbf{1}}}^{[[a],[b]]} = \prod_{i=1}^r \delta_{\ell_i, \tilde{\ell}_i}. \quad (2.54)$$

This renders (2.51) trivially modular covariant.

⁷There is an analogous discussion on modular invariance in [18].

(ii) **All k_i 's are even.** The situation is more involved in this case. We now have $N \in 2\mathbb{Z}$. We define,

$$\begin{aligned} S_1 &= \left\{ i \in \{1, \dots, r\} ; \frac{N}{k_i + 2} \in 2\mathbb{Z} + 1 \right\}, \\ S_2 &= \left\{ i \in \{1, \dots, r\} ; \frac{N}{k_i + 2} \in 2\mathbb{Z} \right\}. \end{aligned} \tag{2.55}$$

One finds that, in addition to (2.53), terms like

$$\prod_i \chi_{\ell_i, [0,1]}^{k_i}(\tau) \cdot \prod_{i \in S_2} \overline{\chi_{\ell_i, [0,1]}^{k_i}(\tau)} \prod_{i \in S_1} \overline{\chi_{\ell_i, [0,1]}^{k_i - \ell_i}(\tau)} \tag{2.56}$$

also contribute (they appear as the $n = N/2$ component in the orbit (2.52)). We thus obtain

$$N_{\mathbf{1}, \bar{\mathbf{1}}}^{[0,1]} = \prod_{i \in S_2} \delta_{\ell_i, \bar{\ell}_i} \prod_{i \in S_1} \left(\delta_{\ell_i, \bar{\ell}_i} + \delta_{\ell_i, k_i - \bar{\ell}_i} \right), \tag{2.57}$$

and by taking the modular transformations, also find

$$N_{\mathbf{1}, \bar{\mathbf{1}}}^{[1,0]} = N_{\mathbf{1}, \bar{\mathbf{1}}}^{[1,1]} = \left(1 + (-1)^{\sum_{i \in S_1} \ell_i} \right) \prod_{i=1}^r \delta_{\ell_i, \bar{\ell}_i}. \tag{2.58}$$

To check the modular covariance we further have to classify

(ii)-(a): $N \in 4\mathbb{Z}$

In this case⁸ we can prove that (1) $S_1 \neq \emptyset$, (2) $\#S_1 \in 2\mathbb{Z}$, (3) $k_i \in 4\mathbb{Z} + 2$ for $\forall i \in S_1$.

(ii)-(b): $N \in 4\mathbb{Z} + 2$

This time we have (1) $S_1 \neq \emptyset$, (2) $k_i \in 4\mathbb{Z}$ for $\forall i \in S_1$.

Making use of these properties and the modular transformation formulas (B.9), one can confirm that (2.57), (2.58) assure the modular covariance of (2.51).

(3) **The free fermion part:** The free fermion part of the flat space-time (transverse part of $\mathbb{R}^{4,1}$) and the S^1 -direction consists of four fermions. As shown in (2.50), the partition sums of the free fermion part $Z_{(a,b)}^f(\tau, \bar{\tau})$ are given as

$$Z_{(a,b)}^f(\tau, \bar{\tau}) = \begin{cases} \left| \left(\frac{\theta_3}{\eta} \right)^2 - (-1)^{\frac{a}{2}} \left(\frac{\theta_4}{\eta} \right)^2 \right|^2, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \left| \left(\frac{\theta_3}{\eta} \right)^2 - (-1)^{\frac{b}{2}} \left(\frac{\theta_2}{\eta} \right)^2 \right|^2, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \left| \left(\frac{\theta_4}{\eta} \right)^2 + i(-1)^{\frac{a+b}{2}} \left(\frac{\theta_2}{\eta} \right)^2 \right|^2, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1). \end{cases} \tag{2.59}$$

⁸Both $N \in 4\mathbb{Z}$ and $N \in 4\mathbb{Z} + 2$ are possible for the $K3$ Gepner model, in contrast to the CY_3 case where we only have the first possibility $N \in 4\mathbb{Z}$ when all k_i are even [18].

One may identify, for instance in the $a \in 2\mathbb{Z}$, $b \in 2\mathbb{Z} + 1$ sector, $\left(\frac{\theta_3}{\eta}\right)^2$ is the NS contribution, while $(-1)^{\frac{a}{2}} \left(\frac{\theta_4}{\eta}\right)^2$ lies in the $\widetilde{\text{NS}}$ sector. The absence of R sector is due to the structure of the chiral blocks in the $K3$ -sector (see table 1). The terms in the other sectors may be identified similarly. It should be remarked that the *winding dependent* phase factors $(-1)^{\frac{a}{2}}$, $(-1)^{\frac{b}{2}}$ and $i(-1)^{\frac{a+b}{2}}$ are necessary for the expected modular covariance. Indeed, *with these phase factors* $Z_{(a,b)}^f(\tau, \bar{\tau})$ behave covariantly under the modular transformations,

$$Z_{(a,b)}^f(-1/\tau, -1/\bar{\tau}) = Z_{(b,-a)}^f(\tau, \bar{\tau}), \quad Z_{(a,b)}^f(\tau + 1, \bar{\tau} + 1) = Z_{(a,a+b)}^f(\tau, \bar{\tau}) \quad (2.60)$$

This can be checked by rewriting the functions $Z_{(a,b)}^f(\tau, \bar{\tau})$ in a unified manner,

$$Z_{(a,b)}^f(\tau, \bar{\tau}) = |G_{(a,b)}(\tau)|^2, \quad G_{(a,b)}(\tau) \equiv 2q^{\frac{a^2}{8}} e^{\frac{i\pi}{4}ab} \left(\frac{\theta_1(\tau, \frac{a\tau+b}{4})}{\eta(\tau)}\right)^4, \quad (2.61)$$

from which the modular covariance immediately follows.

Assembling these results the total partition function $Z(\tau, \bar{\tau}) = Z^u(\tau, \bar{\tau}) + Z^t(\tau, \bar{\tau})$ is indeed verified to be modular invariant with the above coefficients $N_{\mathbf{1}, \mathbf{1}}^{[S, T]}$.

We conclude this section with several comments.

- By our construction $\sigma_{\text{mirror}, R}$ yields the automorphism $\sigma_{(i)}^{\mathcal{N}=2}$ in each $\mathcal{N} = 2$ minimal sector M_{k_i} , whereas $\sigma_{\text{mirror}, L}$ does *not* induce any automorphism in M_{k_i} . We also point out that the transformation interpolating between $\sigma_3^{\mathcal{N}=4}$ and $\sigma_1^{\mathcal{N}=4}$ is generically just an *outer* automorphism of $\mathcal{N} = 4$ SCA. Thus there is no self-evident principle *a priori* that relates the action of $\sigma_3^{\mathcal{N}=4}$ with that of $\sigma_1^{\mathcal{N}=4}$ on the $\mathcal{N} = 4$ *primary states*.
- As addressed above, the chiral blocks of the left-mover have formally the same forms as the right-mover. This means that if we only look at the closed string spectrum the model is indistinguishable from the symmetric orbifold with \mathbb{Z}_2 -twisting $\sigma_L = \sigma_R$, as the closed string partition functions are equal. Nevertheless, it should be emphasised that the asymmetric orbifold (with twist $\sigma_{\text{mirror}, L} \neq \sigma_{\text{mirror}, R}$) is different from the corresponding symmetric orbifold; the distinction being crucial for the physics of D-branes in this string vacuum. As observed in [6], an asymmetric orbifold generally yields different spectrum of geometric D-branes realized by linear gluing conditions from that of a symmetric type orbifold. We also point out that the mirror-involution σ_{mirror} given above still includes a phase ambiguity due to an ambiguity of $\sigma^{\mathcal{N}=2, (i)}$ in each M_{k_i} sector. This phase ambiguity does not affect the torus partition function we have obtained, but it would become important when we examine the D-brane spectrum. We hope to report on detailed aspects on D-branes in mirrorfolds elsewhere.
- These mirrorfold string vacua break the space-time SUSY completely, yielding a non-vanishing cosmological constant at the one-loop level. In fact, as is seen in (2.50),

the partition function in the twisted sector does not vanish at all, in contrast to the untwisted sector which is kept supersymmetric. In other words, the space-time SUSY is broken by the winding string modes, which is similar to the Scherk-Schwarz type S^1 -compactification [15]. It is easy to see that the most tachyonic winding mode appears in the sector of $a = 1$, which has a mass squared proportional to⁹

$$\begin{aligned}
 h - \frac{1}{2} &= -\frac{1}{2} + \min_{\ell_1, \dots, \ell_r} \left[\sum_{i=1}^r h_{\ell_i}^t \right] + \frac{R^2}{4} \\
 &\geq -\frac{1}{4} + \frac{R^2}{4},
 \end{aligned}
 \tag{2.62}$$

where $h_{\ell_i}^t \equiv \frac{k_i - 2 + (k_i - 2\ell_i)^2}{16(k_i + 2)} + \frac{1}{16}$ are the conformal weights of the twisted characters $\chi_{\ell_i, [1,0]}^{k_i}(\tau)$ (B.8). (The minimum value of $h_{\ell_i}^t$ is achieved when $\ell_i = \left\lfloor \frac{k_i}{2} \right\rfloor$, and the inequality (2.62) is saturated iff all the levels k_i are even.) Thus there is no tachyonic instability as long as $R > 1$ (self-dual radius). Of course, since R is a closed string modulus, the base circle may shrink to $R < 1$ and in that case we encounter an instability due to the winding tachyon condensation.

- The modular invariant constructed above is, contrary to what would naively be anticipated, *not* of an order 2 orbifold but rather of an order 4 orbifold. This arises from the fact that the free fermion part $Z_{(a,b)}^f(\tau, \bar{\tau})$ (2.59) is \mathbb{Z}_4 -periodic with respect to the windings a, b , rather than \mathbb{Z}_2 . This is related to the *chiral* spin structures that are characteristic to the type II vacua. If instead considering the type 0 vacua, the free fermion part would take a simpler form

$$Z^{f, \text{type } 0}(\tau, \bar{\tau}) \propto \frac{1}{2} \left[\left| \frac{\theta_3}{\eta} \right|^4 + \left| \frac{\theta_4}{\eta} \right|^4 + \left| \frac{\theta_2}{\eta} \right|^4 \right] \left(\equiv Z_{\text{SO}(4)_1}(\tau, \bar{\tau}) \right), \tag{2.63}$$

which has no dependence on the windings a, b . Consequently, the type 0 vacua of the mirror-folds are realized as order 2 orbifolds as expected from the intuitive picture.

- In the above construction we have chosen the first candidate (2.19) of the left action. It is of course natural to ask what would happen if we instead use the second candidate of operation (2.20), which might give rise to an asymmetric modular invariant, if consistent at all. We were not able to construct a modular invariant using (2.20) for compact $K3$ fibrations, and while we have not exhausted all possibilities, such a construction seems quite unlikely. For instance, in the $a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1$ sector, the partition function typically includes terms like

$$\sum_{\mathbf{1}, \bar{\mathbf{1}}} N_{\mathbf{1}, \bar{\mathbf{1}}}^{[0,1]} \left(\delta_{\mathbf{1}, \mathbf{0}} \text{ch}_0^{\mathcal{N}=4, (\text{NS})}(\ell = 0; \tau) + \sum_{n \geq 1} a_{n, \mathbf{1}} \text{ch}^{\mathcal{N}=4, (\text{NS})}(p_{n, \mathbf{1}}; \tau) \right) \cdot \overline{\sum_{n \geq 0} a_{n, \bar{\mathbf{1}}} \chi_{[0,1]}(p_{n, \bar{\mathbf{1}}}; \tau)}. \tag{2.64}$$

⁹The temporal winding b may be dualized to the KK momentum by Poisson resummation in the standard way, and it does not play any important role in this context. We also find that the sectors of $a \in 4\mathbb{Z} + 2, b \in 2\mathbb{Z} + 1$ include other candidates of winding tachyons because of the wrong GSO projection due to the phase factors appearing in (2.59). However, they are found to be always less tachyonic than that of the $a = 1$ sector and do not alter the discussion here.

The first term in the left moving part is the $\mathcal{N} = 4$ massless character of spin 0 (graviton character), while the second term consists only of the massive characters. Due to an involved behavior of the massless character under the modular S-transformation, the modular invariance of the total partition function is likely to be spoiled (see appendix A).

3. Mirrorfolds with non-compact K3 fibrations

In this section we discuss an extension of the mirrorfold model to include *non-compact* Gepner-like models in the K3-fiber. In contrast to the compact fiber case, both (2.19) and (2.20) are found to be compatible with modular invariance.

3.1 Non-SUSY vacua: symmetric modular invariants

Let us first consider orbifolding by the twist (2.19) as in the previous section. We now assume non-compact Gepner-like models for the K3-fiber, defined by

$$\mathcal{M}_{\text{fiber}} \equiv [M_{k_1} \otimes \cdots \otimes M_{k_r} \otimes L_{\bar{N}, \bar{K}}] / \mathbb{Z}_N, \quad (3.1)$$

$$N \equiv \text{L.C.M.} \{k_i + 2, \bar{N}\},$$

$$\sum_{i=1}^r \frac{k_i}{k_i + 2} + \left(1 + \frac{2\bar{K}}{\bar{N}}\right) = 2, \quad (3.2)$$

where $L_{\bar{N}, \bar{K}}$ denotes the $\text{SL}(2; \mathbb{R}) / \text{U}(1)$ Kazama-Suzuki supercoset model at level $k \equiv \bar{N} / \bar{K}$ (for simplicity we assume \bar{N} and \bar{K} to be relatively prime hereafter). Note, in particular, that this includes the A_{N-1} -type ALE spaces [21] as a simplest case of the fiber SCFT,

$$\mathcal{M}_{\text{fiber}} = [M_{N-2} \otimes L_{N,1}] / \mathbb{Z}_N .$$

With these fibre models we may construct mirrorfolds in the same way as in the previous section,

$$\frac{\mathcal{M}_{\text{fiber}} \times S_{2R}^1}{\sigma_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}} .$$

Now let us work on the partition function. The total partition function generically has the twisted and untwisted parts,

$$Z(\tau, \bar{\tau}) = Z^{\text{u}}(\tau, \bar{\tau}) + Z^{\text{t}}(\tau, \bar{\tau}), \quad (3.3)$$

where we again define the twisted sector $Z^{\text{t}}(\tau, \bar{\tau})$ as including temporal or spatial twist by $\sigma_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}$. The untwisted sector $Z^{\text{u}}(\tau, \bar{\tau})$ involves no such twist. We shall discuss each sector separately.

The untwisted sector. The partition function $Z^{\text{u}}(\tau, \bar{\tau})$ of the untwisted sector is known to be IR-divergent, reflecting the infinite volume of the non-compact target space. The regularized partition function splits into two parts [12] (see also [19, 20]),

$$Z^{\text{u}}(\tau, \bar{\tau}) = Z_{\text{con}}^{\text{u}}(\tau, \bar{\tau}) + Z_{\text{dis}}^{\text{u}}(\tau, \bar{\tau}), \quad (3.4)$$

where the first term includes continuous representations of $L_{\bar{N},\bar{K}}$ and is expanded only with the $\mathcal{N} = 4$ massive characters. This is manifestly modular invariant and is proportional to the (regularized) volume factor $V \sim \ln \epsilon$ (ϵ is the IR cut-off). The continuous part describes the propagating degrees of freedom in the non-compact $K3$ space. The second term $Z_{\text{dis}}^{\text{u}}(\tau, \bar{\tau})$, on the other hand, includes discrete representations of $L_{\bar{N},\bar{K}}$. There is no volume factor in the second term as it corresponds to the localized degrees of freedom around isolated singularities in the background. The $\mathcal{N} = 4$ character expansion of the second (discrete) part involves both the massless *matter* characters (i.e. $\ell = 1/2$ for the NS sector) and the massive characters, but *no graviton (identity) character*. This means that gravity decouples in the string vacua. A potential problem is that $Z_{\text{dis}}^{\text{u}}(\tau, \bar{\tau})$ is not modular invariant in general. A way to circumvent this problem is to focus only on the propagating degrees of freedom, by considering the partition function *per unit volume* as discussed in [12];

$$\lim_{V \rightarrow \infty} \frac{Z}{V} = \lim_{V \rightarrow \infty} \frac{Z_{\text{con.}}}{V} . \tag{3.5}$$

Note that the second term Z_{dis} drops after divided by the infinite volume factor V .

The partition function for the untwisted sector is obtained as [11–13]

$$\frac{Z_{\text{con.}}^{\text{u}}(\tau, \bar{\tau})}{V} = \frac{1}{2} \cdot \frac{1}{4N} \sum_{\alpha, \tilde{\alpha}} \sum_{I, \tilde{I}} \epsilon(\alpha) \epsilon(\tilde{\alpha}) N_{I, \tilde{I}} \mathcal{G}_I^{(\alpha)}(\tau) \overline{\mathcal{G}_{\tilde{I}}^{(\tilde{\alpha})}(\tau)} \frac{1}{\tau_2^2 |\eta(\tau)|^8} Z_{2R}(\tau, \bar{\tau}) \left(\frac{\theta_{[\alpha]}}{\eta} \right)^3 \overline{\left(\frac{\theta_{[\tilde{\alpha}]}}{\eta} \right)^3}, \tag{3.6}$$

where the chiral blocks in the NS sector are

$$\mathcal{G}_I^{(\text{NS})}(\tau, z) \equiv \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} q^{\frac{a^2}{2}} y^a \prod_{i=1}^r \text{ch}_{\ell_i, m_i}^{(\text{NS}), k_i}(\tau, z + a\tau + b) \frac{\Theta_{\bar{m}, \bar{N}\bar{K}}(\tau, \frac{z}{N}(z + a\tau + b))}{\eta(\tau)}, \tag{3.7}$$

and those for the other spin structures are obtained by the $1/2$ spectral flows. Here $I \equiv \{(\ell_i, m_i), \bar{m}\}$ is the collective index. The right-moving chiral blocks are similar, and $N_{I, \tilde{I}}$ are some coefficients (not specified explicitly here)¹⁰ that are compatible with modular invariance. We used the common abbreviation $\mathcal{G}_I^{(\alpha)}(\tau) \equiv \mathcal{G}_I^{(\alpha)}(\tau, 0)$. The $L_{\bar{N},\bar{K}}$ -sector yields additional free oscillator contributions $\frac{1}{|\eta|^2} \left(\frac{\theta_{[\alpha]}}{\eta} \right) \overline{\left(\frac{\theta_{[\tilde{\alpha}]}}{\eta} \right)}$, and the integral over the zero-mode momentum of the non-compact boson (‘Liouville mode’) generates one more factor $\tau_2^{-1/2}$. As addressed above, the partition function $Z_{\text{con.}}^{\text{u}}$ includes only the continuous representations in the $L_{\bar{N},\bar{K}}$ -sector, and its modular invariance is manifest.

¹⁰Note that the quantum numbers \bar{m} in the $L_{\bar{N},\bar{K}}$ -sector need not be symmetric. A typical modular invariant includes

$$\bar{m} = \bar{K}n_0 + \bar{N}w_0, \quad \tilde{\bar{m}} = \bar{K}n_0 - \bar{N}w_0, \quad (n_0 \in \mathbb{Z}_{\bar{N}}, w_0 \in \mathbb{Z}_{2\bar{K}}) .$$

See e.g. [13] for more details.

The twisted sector. The construction of the twisted sector partition function is similar to the compact case. We find,

$$\begin{aligned}
 \frac{Z^t(\tau, \bar{\tau})}{V} &= \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z} \\ b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^2 |\eta|^8} \sum_1 \chi_{1,[0,1]}^{\mathbf{k}}(\tau) \overline{\chi_{1,[0,1]}^{\mathbf{k}}(\tau)} \frac{\theta_3 \theta_4}{\eta^2} \cdot \sqrt{\frac{2\eta}{\theta_2}} \sqrt{\frac{\theta_3 \theta_4}{\eta^2}} \cdot Z_{(a,b)}^f(\tau, \bar{\tau}) \\
 &+ \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ b \in 2\mathbb{Z}}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^2 |\eta|^8} \sum_1 \chi_{1,[1,0]}^{\mathbf{k}}(\tau) \overline{\chi_{1,[1,0]}^{\mathbf{k}}(\tau)} \frac{\theta_2 \theta_3}{\eta^2} \cdot \sqrt{\frac{2\eta}{\theta_4}} \sqrt{\frac{\theta_2 \theta_3}{\eta^2}} \cdot Z_{(a,b)}^f(\tau, \bar{\tau}) \\
 &+ \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^2 |\eta|^8} \sum_1 \chi_{1,[1,1]}^{\mathbf{k}}(\tau) \overline{\chi_{1,[1,1]}^{\mathbf{k}}(\tau)} \frac{\theta_4 \theta_2}{\eta^2} \cdot \sqrt{\frac{2\eta}{\theta_3}} \sqrt{\frac{\theta_4 \theta_2}{\eta^2}} \cdot Z_{(a,b)}^f(\tau, \bar{\tau}).
 \end{aligned}
 \tag{3.8}$$

where we set $[a] \in \mathbb{Z}_2$, $a \equiv [a] \pmod{2}$ as before. We used an abbreviated notation $\chi_1^{\mathbf{k}}(\tau) \equiv \prod_{i=1}^r \chi_{\ell_i}^{k_i}(\tau)$ and $Z_{(a,b)}^f(\tau, \bar{\tau})$ is defined in (2.59).

Note that $\chi_1^{\mathbf{k}}(\tau)$ here plays the same role as the function $f_1^{\mathbf{k}}(\tau)$ in the compact case. Namely, it appears as the trace over the $\mathcal{N} = 4$ primary states. Again we have an additional contribution of $1/\tau_2^{1/2} |\eta|^2$ from the non-compact boson along the linear dilaton direction. Another difference from the compact case is the modular invariant coefficients

$$N_{1,\bar{1}}^{[0,1]} = N_{1,\bar{1}}^{[1,0]} = N_{1,\bar{1}}^{[1,1]} = \prod_{i=1}^r \delta_{\ell_i, \bar{\ell}_i}. \tag{3.9}$$

In the compact case the conformal blocks are related by formulas like

$$\text{ch}_{\ell,m}^{(\text{NS}),k}(\tau, z) = \text{ch}_{k-\ell, m+k+2}^{(\text{NS}),k}(\tau, z),$$

due to the field identification of the minimal models. In the non-compact case such a relation is absent for the conformal blocks of the $L_{\bar{N}, \bar{K}}$ -sector, $\propto q^* \Theta_{\bar{m}, \bar{N}\bar{K}} \frac{\theta_{[\alpha]}}{\eta^3}$, giving rise to the relatively simple coefficients (3.9).

Here we would like to give some comments on the non-compact mirrorfold model.

- As in the compact fiber case, these string vacua are not supersymmetric, and we can likewise examine the tachyonic instability. A slight difference from the compact case is the existence of mass gap $\frac{\bar{K}}{4\bar{N}}$. We find

$$h - \frac{1}{2} = -\frac{1}{2} + \min_{\ell_1, \dots, \ell_r} \left[\sum_{i=1}^r h_{\ell_i}^t \right] + \left(\frac{1}{8} + \frac{\bar{K}}{4\bar{N}} \right) + \frac{R^2}{4}. \tag{3.10}$$

Using the criticality condition (3.2), we again reach the evaluation

$$h - \frac{1}{2} \geq -\frac{1}{4} + \frac{R^2}{4}. \tag{3.11}$$

Therefore, we have no tachyonic instability as long as $R > 1$. However, there exists a crucial difference from the compact case: now the graviton modes are decoupled from

the physical Hilbert space, implying that the radius R becomes non-normalizable. Thus we should regard it as a parameter of the theory rather than a dynamical modulus. There could still exist normalizable closed string moduli corresponding to the massless matter rep. $\mathcal{D}_{1/2}^{(\text{NS})}$ of $\mathcal{N} = 4$ SCA (see [12, 13] for more details). However, the corresponding marginal deformations do not affect the mass square of the winding tachyon (3.11), because they must preserve the $\mathcal{N} = 4$ superconformal symmetry. We thus conclude that these non-supersymmetric string vacua are stable at the level of perturbative string, as long as R is chosen to be greater than the self-dual radius.

- For the simplest case $\mathcal{M}_{\text{fiber}} = [M_{N-2} \otimes L_{N,1}]/\mathbb{Z}_N$, which describes the ALE space of A_{N-1} -type [21], we obtain

$$\frac{Z^u(\tau, \bar{\tau})}{V} = \frac{1}{4} \sum_{\alpha, \bar{\alpha}} \epsilon(\alpha) \epsilon(\bar{\alpha}) Z_{2R}(\tau, \bar{\tau}) Z_{\text{SU}(2)_k}(\tau, \bar{\tau}) \cdot \frac{1}{\tau_2^2 |\eta|^8} \left(\frac{\theta_{[\alpha]}}{\eta} \right)^4 \overline{\left(\frac{\theta_{[\bar{\alpha}]}}{\eta} \right)^4}, \quad (3.12)$$

$$\frac{Z^t(\tau, \bar{\tau})}{V} = \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ \text{or } b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^2 |\eta|^8} \sum_{\ell} \chi_{\ell, [[a],[b]]}^k(\tau) \overline{\chi_{\ell, [[a],[b]]}^k(\tau)} \widehat{Z}_{(a,b)}^f(\tau, \bar{\tau}), \quad (3.13)$$

where the free fermion part is written as

$$\widehat{Z}_{(a,b)}^f(\tau, \bar{\tau}) = \begin{cases} \left| \left(\frac{\theta_3}{\eta} \right)^3 \frac{\theta_4}{\eta} - (-1)^{\frac{a}{2}} \left(\frac{\theta_4}{\eta} \right)^3 \frac{\theta_3}{\eta} \right|^2, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \left| \left(\frac{\theta_3}{\eta} \right)^3 \frac{\theta_2}{\eta} - (-1)^{\frac{b}{2}} \left(\frac{\theta_2}{\eta} \right)^3 \frac{\theta_3}{\eta} \right|^2, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \left| \left(\frac{\theta_4}{\eta} \right)^3 \frac{\theta_2}{\eta} + i(-1)^{\frac{a+b}{2}} \left(\frac{\theta_2}{\eta} \right)^3 \frac{\theta_4}{\eta} \right|^2, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1). \end{cases} \quad (3.14)$$

To derive (3.12) we have used the familiar branching relation for the $\mathcal{N} = 2$ minimal characters (A.5), (A.8). We also note

$$\sum_{\ell} \chi_{\ell, [[a],[b]]}^k(\tau) \overline{\chi_{\ell, [[a],[b]]}^k(\tau)} = \sum_{\ell} \chi_{\ell, (a,b)}^{\text{SU}(2)_k}(\tau) \overline{\chi_{\ell, (a,b)}^{\text{SU}(2)_k}(\tau)}, \quad (3.15)$$

where the R.H.S is written in terms of the twisted $\text{SU}(2)_k$ characters (C.1).¹¹ This is consistent with the fact that σ_{mirror} is now interpretable as the $(e^{i\pi K_0^3}, e^{i\pi \bar{K}_0^1})$ -twisting in the $\text{SU}(2)$ supersymmetric WZW model of level $N \equiv k+2$, where K^a are the (total) $\text{SU}(2)$ -currents, if recalling [21]

$$[M_{N-2} \otimes L_{N,1}]/\mathbb{Z}_N \cong \mathbb{R}_\phi \times \text{SU}(2)_N.$$

Obviously, the model defined by (3.12), (3.13) is regarded as a supersymmetric analogue of the $\text{SU}(2)$ T-fold considered in [6]. In this case the interpolation between $\sigma_1^{\mathcal{N}=4}$ and $\sigma_3^{\mathcal{N}=4}$ is exceptionally realized as an inner automorphism.

¹¹It is important to notice that $|\chi_{\ell, (a,b)}^{\text{SU}(2)_k}(\tau)|^2$ is \mathbb{Z}_2 -periodic with respect to a, b , even though the chiral part $\chi_{\ell, (a,b)}^{\text{SU}(2)_k}(\tau)$ breaks that periodicity due to an extra phase factor. Therefore, (3.15) is a consistent relation.

3.2 SUSY vacua: asymmetric modular invariants

In contrast to the compact fiber case, the second candidate of the mirror-involution (2.20) turns out to yield consistent mirrorfolds, as we shall demonstrate below. We denote the involution of (2.20) as $\widehat{\sigma}_{\text{mirror}}$ in order to distinguish it from the first one. Since $\widehat{\sigma}_{\text{mirror}}$ acts on the $\mathcal{N} = 4$ SCA as $(\mathbf{1}, \sigma_{1,R}^{\mathcal{N}=4})$, the resultant partition function will provide an asymmetric modular invariant. What differs crucially from the compact models is that we include *only* the massive representations. The massive characters possess simpler modular properties that makes an asymmetric modular invariant possible.

The model is described as follows. The untwisted sector has the same partition function (3.6). The partition function in the twisted sector is given as

$$Z^t(\tau, \bar{\tau}) = \frac{1}{4} \sum_{\substack{a \in 2\mathbb{Z}+1 \\ \text{or } b \in 2\mathbb{Z}+1}} Z_{R,(a,b)}(\tau, \bar{\tau}) \frac{1}{\tau_2^2 |\eta|^8} \sum_1 \chi_{1,[[a],[b]]}^{\mathbf{k}}(\tau) \overline{\chi_{1,[[a],[b]]}^{\mathbf{k}}(\tau)} \widehat{Z}_{(a,b)}^{f,\text{SUSY}}(\tau, \bar{\tau}). \quad (3.16)$$

The free fermion part is now written as

$$\widehat{Z}_{(a,b)}^{f,\text{SUSY}}(\tau, \bar{\tau}) = \left[\left(\frac{\theta_3}{\eta} \right)^4 - \left(\frac{\theta_4}{\eta} \right)^4 - \left(\frac{\theta_2}{\eta} \right)^4 \right] \cdot \overline{\left(\frac{\theta \cdot \theta_{[a,b]}}{\eta^2} \right)} G_{(a,b)}(\tau), \quad (3.17)$$

where $G_{(a,b)}(\tau)$ is defined in (2.61), and we used abbreviation,

$$\frac{\theta \cdot \theta_{[a,b]}}{\eta^2} \equiv \begin{cases} \frac{\theta_3 \theta_4}{\eta^2}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ \frac{\theta_2 \theta_3}{\eta^2}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ \frac{\theta_4 \theta_2}{\eta^2}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1). \end{cases} \quad (3.18)$$

More explicitly,

$$\begin{aligned} & \widehat{Z}_{(a,b)}^{f,\text{SUSY}}(\tau, \bar{\tau}) \\ = & \begin{cases} e^{\frac{i\pi}{4}ab} \left[\left(\frac{\theta_3}{\eta} \right)^4 - \left(\frac{\theta_4}{\eta} \right)^4 - \left(\frac{\theta_2}{\eta} \right)^4 \right] \cdot \overline{\left[\left(\frac{\theta_3}{\eta} \right)^3 \frac{\theta_4}{\eta} - (-1)^{\frac{a}{2}} \left(\frac{\theta_4}{\eta} \right)^3 \frac{\theta_3}{\eta} \right]}, & (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1), \\ e^{-\frac{i\pi}{4}ab} \left[\left(\frac{\theta_3}{\eta} \right)^4 - \left(\frac{\theta_4}{\eta} \right)^4 - \left(\frac{\theta_2}{\eta} \right)^4 \right] \cdot \overline{\left[\left(\frac{\theta_3}{\eta} \right)^3 \frac{\theta_2}{\eta} - (-1)^{\frac{b}{2}} \left(\frac{\theta_2}{\eta} \right)^3 \frac{\theta_3}{\eta} \right]}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}), \\ -e^{-\frac{i\pi}{4}ab} \left[\left(\frac{\theta_3}{\eta} \right)^4 - \left(\frac{\theta_4}{\eta} \right)^4 - \left(\frac{\theta_2}{\eta} \right)^4 \right] \cdot \overline{\left[\left(\frac{\theta_4}{\eta} \right)^3 \frac{\theta_2}{\eta} + i(-1)^{\frac{a+b}{2}} \left(\frac{\theta_2}{\eta} \right)^3 \frac{\theta_4}{\eta} \right]}, & (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} + 1). \end{cases} \end{aligned} \quad (3.19)$$

At first glance, (3.19) might appear inconsistent with unitarity due to phase factors depending on winding numbers a, b . In a unitary theory the torus partition function has to be real when $\tau = i\tau_2$ (i.e. $\text{Re } \tau = 0$). We note that, when $\tau = i\tau_2$,

$$\overline{Z_{R,(a,b)}(\tau, \bar{\tau})} = Z_{R,(a,-b)}(\tau, \bar{\tau}), \quad \overline{\widehat{Z}_{(a,b)}^{f,\text{SUSY}}(\tau, \bar{\tau})} = \widehat{Z}_{(a,-b)}^{f,\text{SUSY}}(\tau, \bar{\tau}), \quad (3.20)$$

so the total partition function is indeed real when $\tau = i\tau_2$, after summing over a, b . Hence there is no inconsistency with unitarity.

In these models the left-movers are expanded by the $\mathcal{N} = 4$ massive characters *with no twisting*, which contribute as $\left(\frac{\theta_{[\alpha]}}{\eta} \right)^2$ in the partition sum. The right-moving chiral

blocks are twisted by $\sigma_1^{\mathcal{N}=4}$, so the models are interpreted as mirrorfolds. Recall that the twisted characters contribute to the free fermion part as $\frac{\theta \cdot \theta_{[a,b]}}{\eta^2}$. The space-time SUSY is achieved by the standard GSO projection acting only on the left-mover, which preserves 8 supercharges.

The model is free from any tachyonic instability in these supersymmetric mirrorfolds, as it should. If we only look at the right-mover, it might seem possible to have winding tachyon modes (belonging to the $a \in 2\mathbb{Z} + 1$ -sectors), similarly to the previous argument in the compact case. However, this does not happen because such string excitations never satisfy the level matching condition and the physical Hilbert space does not include them.

3.3 Comments on D-branes: breakdown of the space-time SUSY

Finally, we mention some interesting features of D-branes in these supersymmetric mirrorfolds, although detailed studies on D-branes will be left to our future work.

A remarkable fact is that *all D-branes in these string vacua are non-BPS*. Recall that the space-time supercharges only come from the left-mover, so no boundary state can preserve the space-time SUSY. In other words, adding any D-brane breaks the space-time SUSY completely.

A typical boundary state describing a D-brane in these vacua has the form

$$|B\rangle = \frac{1 + \hat{\sigma}_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}}{\sqrt{2}} |B\rangle_0, \tag{3.21}$$

where $|B\rangle_0$ is a boundary state in the ‘parent theory’ $K3 \times S^1_{2R}$. As just mentioned, adding this brane breaks the space-time SUSY, so we expect to have open string tachyons which would lead to an IR instability of this vacuum.

Let us briefly discuss whether the cylinder amplitude such as $\langle B | e^{-\pi s H^{(c)}} | B \rangle$ ($H^{(c)}$ is the closed string Hamiltonian) gives rise to an IR instability. After taking account of the contribution from the flat space-time and summing over spin structures, the term with no insertion of $\hat{\sigma}_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}$ provides a vanishing open string amplitude, because the GSO projection correctly acts on it. However, this is not the case for the term in which $\hat{\sigma}_{\text{mirror}} \otimes \mathcal{T}_{2\pi R}$ is inserted, due to the lack of GSO projection in the open string channel. It is not difficult to see that the NS sector yields the leading contribution to the non-SUSY piece of the open channel amplitude. It would look like ($q = e^{-2\pi t}$, $t \equiv 1/s$)

$$Z^{\text{cyl, (NS)}}(it) \sim \int_0^\infty dp \sum_I \sum_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \rho_I(p) c_{I,n} q^{\frac{p^2}{2} + h_I + \frac{\bar{K}}{4N} + n - \frac{1}{8}} \frac{2}{\theta_4(it)} \times [\text{sectors other than } K3], \tag{3.22}$$

with some non-trivial density function¹² $\rho_I(p)$ and coefficients $c_{I,n} \in \mathbb{Z}_{\geq 0}$ determined from the boundary wave function of $|B\rangle_0$. The relevant term contributes to the lightest open string mode as $h_{\text{min}} = \frac{\bar{K}}{4N} + \frac{1}{8}$. Here, the contribution 1/8 is due to the twisted character $\chi_{[1,0]}(p; it) = q^{\frac{p^2}{2}} \frac{2}{\theta_4(it)}$, (with $h = \frac{p^2}{2} + \frac{1}{4}$). When the brane is localized along the base

¹²In the simple case of ALE fiber, the density $\rho_I(p)$ is explicitly calculated in [22]. See also [23].

circle, we also have winding energy of open strings R^2 originating from the $\mathcal{T}_{2\pi R}$ insertion, whereas no more contribution when the brane is wrapped around the base.

To summarise,

- **D-branes localized along the base:** The open string mass squared behaves as

$$h - \frac{1}{2} \geq \frac{\bar{K}}{4\bar{N}} - \frac{3}{8} + R^2 . \tag{3.23}$$

Hence the vacuum is IR stable as long as $R > R_c \equiv \sqrt{\frac{3}{8} - \frac{\bar{K}}{4\bar{N}}}$, whereas unstable if $R < R_c$.

- **D-branes wrapped around the base:** The open string mass squared behaves as

$$h - \frac{1}{2} \geq \frac{\bar{K}}{4\bar{N}} - \frac{3}{8} . \tag{3.24}$$

The vacuum is always IR unstable. (Note that $\frac{\bar{K}}{4\bar{N}} \leq \frac{1}{8}$ holds because of the criticality condition (3.2).)

Again R is not a normalizable modulus, and any normalizable moduli inherited from both closed and open string modes do not affect the above evaluation of the lightest open string mass.

4. Discussions

In this paper we have studied a class of non-geometric backgrounds of superstring theory defined with the twisting by the mirror transformation on a $K3$ space which we call ‘mirrorfolds’. We have mainly elaborated on how we can construct modular invariant models that describe mirrorfolds. We have also discussed possible instability caused by winding tachyon condensations. To achieve modular invariance, it has been crucial to carefully fix the action of the mirror-involution on the $\mathcal{N} = 4$ primary states.

It would be a little surprising that we have several significant distinctions between the compact and the non-compact models. As we have demonstrated, supersymmetric mirrorfolds can exist only in the non-compact models in which gravity decouples. We have also found that the compact mirrorfolds are always unstable due to the tachyonic modes wound around the base circle. From the viewpoints of representation theory of $\mathcal{N} = 4$ SCA, the difference of these two theories originates from the modular properties of the irreducible characters of the $\mathcal{N} = 4$ SCA. The graviton character, which only appears in the compact models, has complicated modular properties that makes possibility of modular invariance so restricted compared with the non-compact models.

A possible future direction related to the present work would be to study D-branes in these vacua. As we have already mentioned (see the second comment at the end of section 2), the phase ambiguity of σ_{mirror} has not been completely removed. This would be important when working with the D-brane spectrum, although it was immaterial for the construction of modular invariant partition functions. In particular, how the Cardy conditions restrict this phase ambiguity is an interesting issue to study.

It is also interesting to compare the analysis given in this paper with models in which the $K3$ -fibers are realized as orbifolds T^4/Γ , where Γ is some discrete subgroup of $SU(2) \subset SO(4)$ acting on T^4 . As is familiar [16], some of Gepner points are also interpretable as orbifolds of T^4 , and it will be anticipated that the T-fold construction works for those orbifold models. It would be non-trivial, however, whether such a T-folding is equivalent with the ‘mirrorfolding’ argued in this paper, or how these two should be related.

Acknowledgments

Y.S. was partly supported by Ministry of Education, Culture, Sports, Science and Technology of Japan. S.K. acknowledges support from JSPS (Research Fellowship for Young Scientists) and the Academy of Finland (Finnish-Japanese Core Programme).

A. Some conventions and notations

In this appendix we collect formulae frequently used in the paper. We use modular parameters $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi iz}$ and theta functions defined by

$$\begin{aligned} \theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1-yq^m)(1-y^{-1}q^m), \\ \theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1-q^m)(1+yq^m)(1+y^{-1}q^m), \\ \theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}), \\ \theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}). \end{aligned} \tag{A.1}$$

We also use

$$\Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}, \tag{A.2}$$

and the Dedekind function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{A.3}$$

We abbreviate as $\theta_i \equiv \theta_i(\tau, 0)$ ($\theta_1 \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0)$ when no confusion arises. The character of $SU(2)_k$ with spin $\ell/2$ ($0 \leq \ell \leq k$) is

$$\chi_{\ell}^{SU(2)_k}(\tau, z) = \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{-\ell-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)}. \tag{A.4}$$

The branching relation corresponding to the coset construction of the $\mathcal{N} = 2$ minimal models $\frac{SU(2)_k \times U(1)_2}{U(1)_{k+2}}$ is given by

$$\chi_{\ell}^{SU(2)_k}(\tau, w) \Theta_{s,2}(\tau, w-z) = \sum_{m \in \mathbb{Z}_2(k+2)} \chi_m^{\ell,s}(\tau, z) \Theta_{m,k+2}\left(\tau, w - \frac{2z}{k+2}\right). \tag{A.5}$$

Here,

$$\chi_m^{\ell,s}(\tau, z) = \sum_{r \in \mathbb{Z}_k} c_{\ell, m-s+4r}^{(k)}(\tau) \Theta_{2m+(k+2)(-s+4r), 2k(k+2)} \left(\tau, \frac{z}{k+2} \right), \quad (\text{A.6})$$

$s \in \mathbb{Z}_4$, and $c_{\ell, m}^{(k)}(\tau)$ are the level k string functions defined by

$$\chi_\ell^{\text{SU}(2)_k}(\tau, z) = \sum_{m \in \mathbb{Z}_{2k}} c_{\ell, m}^{(k)}(\tau) \Theta_{m, k}(\tau, z). \quad (\text{A.7})$$

The $\mathcal{N} = 2$ minimal model characters are related to $\chi_m^{\ell,s}$ as

$$\begin{aligned} \text{ch}_{\ell, m}^{(\text{NS}), k}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{\ell, m}^{\text{NS}}} q^{L_0 - \hat{c}/8} y^{J_0} = \chi_m^{\ell, 0}(\tau, z) + \chi_m^{\ell, 2}(\tau, z), \\ \text{ch}_{\ell, m}^{(\widetilde{\text{NS}}), k}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{\ell, m}^{\widetilde{\text{NS}}}} (-1)^F q^{L_0 - \hat{c}/8} y^{J_0} = \chi_m^{\ell, 0}(\tau, z) - \chi_m^{\ell, 2}(\tau, z), \\ \text{ch}_{\ell, m}^{(\text{R}), k}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{\ell, m}^{\text{R}}} q^{L_0 - \hat{c}/8} y^{J_0} = \chi_m^{\ell, 1}(\tau, z) + \chi_m^{\ell, -1}(\tau, z), \\ \text{ch}_{\ell, m}^{(\widetilde{\text{R}}), k}(\tau, z) &\equiv \text{Tr}_{\mathcal{H}_{\ell, m}^{\widetilde{\text{R}}}} (-1)^F q^{L_0 - \hat{c}/8} y^{J_0} = \chi_m^{\ell, 1}(\tau, z) - \chi_m^{\ell, -1}(\tau, z). \end{aligned} \quad (\text{A.8})$$

The level 1 (small) $\mathcal{N} = 4$ characters are given as [17]

massive characters:

$$\text{ch}^{\mathcal{N}=4, (\text{NS})}(h; \tau, z) = q^{h - \frac{1}{8}} \frac{\theta_3(\tau, z)^2}{\eta(\tau)^3}, \quad (\text{for } \mathcal{C}_h^{(\text{NS})}). \quad (\text{A.9})$$

massless characters:

$$\begin{aligned} \text{ch}_0^{\mathcal{N}=4, (\text{NS})} \left(\ell = \frac{1}{2}; \tau, z \right) &= q^{-1/8} \sum_{n \in \mathbb{Z}} \frac{1}{1 + yq^{n-1/2}} q^{\frac{n^2}{2}} y^n \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \quad (\text{for } \mathcal{D}_{1/2}^{(\text{NS})}), \quad (\text{A.10}) \\ \text{ch}_0^{\mathcal{N}=4, (\text{NS})}(\ell = 0; \tau, z) &= q^{-1/8} \sum_{n \in \mathbb{Z}} \frac{yq^{n-1/2} - 1}{1 + yq^{n-1/2}} q^{\frac{n^2}{2}} y^n \frac{\theta_3(\tau, z)}{\eta(\tau)^3} \\ &= q^{-1/8} \sum_{n \in \mathbb{Z}} \frac{(1-q)q^{\frac{n^2}{2} + n - \frac{1}{2}} y^{n+1}}{(1 + yq^{n+1/2})(1 + yq^{n-1/2})} \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \quad (\text{for } \mathcal{D}_0^{(\text{NS})}). \end{aligned} \quad (\text{A.11})$$

The following identity is often useful:

$$\text{ch}^{\mathcal{N}=4, (\text{NS})}(h; \tau, z) = q^h \left(\text{ch}_0^{\mathcal{N}=4, (\text{NS})}(\ell = 0; \tau, z) + 2\text{ch}_0^{\mathcal{N}=4, (\text{NS})} \left(\ell = \frac{1}{2}; \tau, z \right) \right). \quad (\text{A.12})$$

An important property of the $\mathcal{N} = 4$ characters is that they decompose into spectrally flowed $\mathcal{N} = 2$ irreducible characters [17],

$$\begin{aligned} \text{ch}^{\mathcal{N}=4, (\text{NS})}(h; \tau, z) &= \sum_{n \in \mathbb{Z}} q^{n^2} y^{2n} \text{ch}^{\mathcal{N}=2, (\text{NS})}(h, Q = 0; \tau, z + n\tau), \\ \text{ch}_0^{\mathcal{N}=4, (\text{NS})} \left(\ell = \frac{1}{2}; \tau, z \right) &= \sum_{n \in \mathbb{Z}} q^{n^2} y^{2n} \text{ch}_M^{\mathcal{N}=2, (\text{NS})}(Q = 1; \tau, z + n\tau), \\ \text{ch}_0^{\mathcal{N}=4, (\text{NS})}(\ell = 0; \tau, z) &= \sum_{n \in \mathbb{Z}} q^{n^2} y^{2n} \text{ch}_G^{\mathcal{N}=2, (\text{NS})}(\tau, z + n\tau), \end{aligned} \quad (\text{A.13})$$

where the three types of $\mathcal{N} = 2$ irreducible characters at $\hat{c} = 2$ are given as

massive characters:

$$\text{ch}^{\mathcal{N}=2,(\text{NS})}(h, Q; \tau, z) = q^{h-\frac{1}{8}} y^Q \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \quad (\text{A.14})$$

massless matter characters:

$$\text{ch}_M^{\mathcal{N}=2,(\text{NS})}(Q; \tau, z) = q^{\frac{|Q|}{2}-\frac{1}{8}} y^Q \frac{1}{1+y^{\text{sgn}(Q)} q^{1/2}} \frac{\theta_3(\tau, z)}{\eta(\tau)^3}, \quad (\text{A.15})$$

graviton character:

$$\text{ch}_G^{\mathcal{N}=2,(\text{NS})}(\tau, z) = q^{-1/8} \frac{(1-q)q^{-1/2}y}{(1+yq^{1/2})(1+yq^{-1/2})} \frac{\theta_3(\tau, z)}{\eta(\tau)^3}. \quad (\text{A.16})$$

The R-sector characters are obtained by the 1/2-spectral flow. Namely,

$$\begin{aligned} \text{ch}^{\mathcal{N}=4,(\text{R})}(h; \tau, z) &= q^{\frac{1}{4}} y \text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h - \frac{1}{4}; \tau, z + \frac{\tau}{2}\right), \quad (\text{for } \mathcal{C}_h^{(\text{R})}), \\ \text{ch}_0^{\mathcal{N}=4,(\text{R})}(\ell; \tau, z) &= q^{\frac{1}{4}} y \text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\frac{1}{2} - \ell; \tau, z + \frac{\tau}{2}\right), \quad (\text{for } \mathcal{D}_\ell^{(\text{R})}). \end{aligned} \quad (\text{A.17})$$

For the convenience of readers we also reproduce the modular transformation formulas of the $\mathcal{N} = 4$ characters at level 1 [17]. We only give the NS sector results as the others are readily obtained by spectral flows.

(i) massive representations

$$\text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h = \frac{p^2}{2} + \frac{1}{8}; -\frac{1}{\tau}, \frac{z}{\tau}\right) = 2e^{i\pi\frac{2z^2}{\tau}} \int_0^\infty dp' \cos(2\pi pp') \text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h = \frac{p'^2}{2} + \frac{1}{8}; \tau, z\right), \quad (\text{A.18})$$

(ii) massless representations

$$\begin{aligned} \text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\ell = 0; -\frac{1}{\tau}, \frac{z}{\tau}\right) &= e^{i\pi\frac{2z^2}{\tau}} \left\{ 2\text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\ell = \frac{1}{2}; \tau, z\right) \right. \\ &\quad \left. + 2 \int_0^\infty dp' \sinh(\pi p') \tanh(\pi p') \text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h = \frac{p'^2}{2} + \frac{1}{8}; \tau, z\right) \right\}, \end{aligned} \quad (\text{A.19})$$

$$\begin{aligned} \text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\ell = \frac{1}{2}; -\frac{1}{\tau}, \frac{z}{\tau}\right) &= e^{i\pi\frac{2z^2}{\tau}} \left\{ -\text{ch}_0^{\mathcal{N}=4,(\text{NS})}\left(\ell = \frac{1}{2}; \tau, z\right) \right. \\ &\quad \left. + \int_0^\infty dp' \frac{1}{\cosh(\pi p')} \text{ch}^{\mathcal{N}=4,(\text{NS})}\left(h = \frac{p'^2}{2} + \frac{1}{8}; \tau, z\right) \right\}. \end{aligned} \quad (\text{A.20})$$

Note the appearance of *both* continuous and discrete terms in the massless formulas (A.19) and (A.20). This feature is characteristic to the massless representations.

B. Twisted characters of $\mathcal{N} = 2$ SCFT

The twisted $\mathcal{N} = 2$ characters are defined with respect to \mathbb{Z}_2 -automorphism of the $\mathcal{N} = 2$ SCA,

$$\sigma^{\mathcal{N}=2} : T \longrightarrow T, \quad J \longrightarrow -J, \quad G^\pm \longrightarrow G^\mp. \quad (\text{B.1})$$

We denote the twisted characters as $\text{ch}_{[S,T]}^{(\alpha)}$, where α are the spin structures, and $S, T \in \mathbb{Z}_2$ signify the spatial and temporal boundary conditions associated with the $\sigma^{\mathcal{N}=2}$ -twist ($S, T = 1$ means twisted, and $S, T = 0$ means no twist). As the $\sigma^{\mathcal{N}=2}$ -twist projects out states with non-vanishing U(1)-charges, the conformal weights are the only quantum numbers relevant in the twisted sectors. It is easy to verify the following identities (see e.g. [18]):

$$\text{ch}_{[0,1]}^{(\text{NS})}(\tau) = \text{ch}_{[0,1]}^{(\widetilde{\text{NS}})}(\tau), \quad \text{ch}_{[1,0]}^{(\text{NS})}(\tau) = \text{ch}_{[1,0]}^{(\text{R})}(\tau), \quad \text{ch}_{[1,1]}^{(\widetilde{\text{NS}})}(\tau) = \text{ch}_{[1,1]}^{(\text{R})}(\tau), \quad (\text{B.2})$$

$$\text{ch}_{[0,1]}^{(\text{R})}(\tau) = \text{ch}_{[0,1]}^{(\widetilde{\text{R}})}(\tau), \quad \text{ch}_{[1,0]}^{(\widetilde{\text{NS}})}(\tau) = \text{ch}_{[1,0]}^{(\widetilde{\text{R}})}(\tau), \quad \text{ch}_{[1,1]}^{(\text{NS})}(\tau) = \text{ch}_{[1,1]}^{(\widetilde{\text{R}})}(\tau). \quad (\text{B.3})$$

We denote the twisted characters in the first line (B.2) as $\chi_{[0,1]}(\tau)$, $\chi_{[1,0]}(\tau)$ and $\chi_{[1,1]}(\tau)$. To find their explicit forms, it is easiest to first evaluate the characters $\chi_{[0,1]} \equiv \text{Tr}[\sigma^{\mathcal{N}=2} q^{L_0 - \frac{\hat{c}}{8}}]$ and then modular transform them to the other boundary conditions. It is obvious that only neutral ($Q = 0$) representations that are invariant under $\sigma^{\mathcal{N}=2}$ -action can contribute to these characters. For any $\mathcal{N} = 2$ SCFT with $\hat{c} > 1$, they are written in simple forms,

$$\begin{aligned} \chi_{[0,1]}(p; \tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_2(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{\hat{c} - 1}{8} \right), \\ \chi_{[1,0]}(p; \tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_4(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{\hat{c}}{8} \right), \\ \chi_{[1,1]}(p; \tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_3(\tau)}, \quad \left(h = \frac{p^2}{2} + \frac{\hat{c}}{8} \right). \end{aligned} \quad (\text{B.4})$$

For the second line (B.3), only the representations that are kept invariant under $\sigma^{\mathcal{N}=2}$ can again contribute to $\text{ch}_{[0,1]}^{(\text{R})}$ (or $\text{ch}_{[0,1]}^{(\widetilde{\text{R}})}$). Most of such representations, however, yield vanishing characters due to fermionic zero-modes. There only exists one exception: the representations generated by Ramond ground states ($h = \frac{\hat{c}}{8}$) with $Q = 0$. In that case, oscillator parts cancel out (as in Witten index), and we simply obtain

$$\text{ch}_{[0,1]}^{(\text{R})} \left(h = \frac{\hat{c}}{8}, Q = 0; \tau \right) \left(= \text{ch}_{[0,1]}^{(\widetilde{\text{R}})} \left(h = \frac{\hat{c}}{8}, Q = 0; \tau \right) \right) = \pm \frac{\prod_{n=1}^{\infty} (1 + q^n)(1 - q^n)}{\prod_{n=1}^{\infty} (1 + q^n)(1 - q^n)} = \pm 1. \quad (\text{B.5})$$

Here we have a sign ambiguity depending on the $\sigma^{\mathcal{N}=2}$ -action on Ramond ground states. The characters of the other boundary conditions in (B.3) are easily obtained by modular transformations; they are merely equal ± 1 .

The twisted characters of the minimal models M_k are more involved. The character formulas corresponding to (B.2) are summarized in [18] (based on [24–27]):

$$\begin{aligned} \chi_{\ell[0,1]}^k(\tau) &= \begin{cases} \frac{2}{\theta_2(\tau)} \left(\Theta_{2(\ell+1),4(k+2)}(\tau) + (-1)^k \Theta_{2(\ell+1)+4(k+2),4(k+2)}(\tau) \right) & (\ell : \text{even}), \\ 0 & (\ell : \text{odd}). \end{cases} \\ \chi_{\ell[1,0]}^k(\tau) &= \frac{1}{\theta_4(\tau)} \left(\Theta_{\ell+1-\frac{k+2}{2},k+2}(\tau) - \Theta_{-(\ell+1)-\frac{k+2}{2},k+2}(\tau) \right) \\ &= \frac{1}{\theta_4(\tau)} \left(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau) + \Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau) \right. \\ &\quad \left. - \Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau) - \Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau) \right), \\ \chi_{\ell[1,1]}^k(\tau) &= \frac{1}{\theta_3(\tau)} \left(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau) + (-1)^k \Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau) \right. \\ &\quad \left. + (-1)^\ell \Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau) + (-1)^{k+\ell} \Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau) \right). \end{aligned} \quad (\text{B.6})$$

The conformal dimensions of the ground states corresponding to the first characters are

$$h = h_\ell \equiv \frac{\ell(\ell+2)}{4(k+2)}, \quad (\text{B.7})$$

(which coincide with those for the $SU(2)_k$ primaries). The ground states of the second and third ones have dimensions

$$h = h_\ell^t \equiv \frac{k-2+(k-2\ell)^2}{16(k+2)} + \frac{1}{16}. \quad (\text{B.8})$$

The states characterised by (B.8) are interpreted as the product of the twist field in the $U(1)$ -sector and the “ C -disorder field” [25] in the \mathbb{Z}_k -parafermion theory [28]. Note that $\chi_{k-\ell[1,0]}^k = \chi_{\ell[1,0]}^k$, $\chi_{k-\ell[1,1]}^k = \chi_{\ell[1,1]}^k$. Due to these relations the corresponding fields are identified, leaving only $\ell = 0, 1, \dots, \lfloor \frac{k}{2} \rfloor$ as independent primary fields.

The modular transformations of the twisted $\mathcal{N} = 2$ characters are

$$\begin{aligned} \chi_{\ell[0,1]}^k(\tau+1) &= e^{2\pi i \left(h_\ell - \frac{k}{8(k+2)} \right)} \chi_{\ell[0,1]}^k(\tau), & \chi_{\ell[0,1]}^k \left(-\frac{1}{\tau} \right) &= \sum_{\ell'=0}^k (-1)^{\ell/2} S_{\ell,\ell'} \chi_{\ell'[1,0]}^k(\tau), \\ \chi_{\ell[1,0]}^k(\tau+1) &= e^{2\pi i \left(h_\ell^t - \frac{k}{8(k+2)} \right)} \chi_{\ell[1,0]}^k(\tau), & \chi_{\ell[1,0]}^k \left(-\frac{1}{\tau} \right) &= \sum_{\ell'=0}^k S_{\ell,\ell'} (-1)^{\ell'/2} \chi_{\ell'[0,1]}^k(\tau), \\ \chi_{\ell[1,1]}^k(\tau+1) &= e^{2\pi i \left(h_\ell^t - \frac{k}{8(k+2)} \right)} \chi_{\ell[1,1]}^k(\tau), & \chi_{\ell[1,1]}^k \left(-\frac{1}{\tau} \right) &= \sum_{\ell'=0}^k \widehat{S}_{\ell,\ell'} \chi_{\ell'[1,1]}^k(\tau). \end{aligned} \quad (\text{B.9})$$

Here $S_{\ell,\ell'} \equiv \sqrt{\frac{2}{k+2}} \sin \left(\frac{\pi(\ell+1)(\ell'+1)}{k+2} \right)$ is the modular S-matrix of the $SU(2)$ WZW model at level k , and $\widehat{S}_{\ell,\ell'} \equiv e^{\frac{\pi i}{2}(\ell+\ell'-\frac{k}{2})} S_{\ell,\ell'}$.

Finally, we mention the remaining minimal model characters appearing in (B.3). In contrast to the $\hat{c} > 1$ case, these characters always vanish. For instance, let us pick up the boundary condition $\{\mathbf{R}, [0, 1]\}$. Only the representations generated by doubly degenerated

primary states $|\ell, m, s\rangle = |\ell, 0, \pm 1\rangle$ ($\ell \in 2\mathbb{Z} + 1$) can contribute, but the trace over them vanishes because $\sigma^{\mathcal{N}=2}$ acts as

$$\sigma^{\mathcal{N}=2} : |\ell, 0, \pm 1\rangle \longmapsto |\ell, 0, \mp 1\rangle .$$

C. Twisted $SU(2)_k$ characters

The twisted characters of the $SU(2)_k$ current algebra are generally written as

$$\chi_{\ell,(a,b)}^{\text{SU}(2)_k}(\tau, z) \equiv e^{2\pi i \frac{k}{4} ab} q^{\frac{k}{4} a^2} y^{\frac{k}{2} a} \chi_{\ell}^{\text{SU}(2)_k}(\tau, z + a\tau + b), \quad (\text{C.1})$$

where a and b parameterize the spatial and temporal boundary conditions. This is a special case of more general formulas for the twisted characters of affine Kac-Moody algebras [29] (up to phase factors). Especially, $(a, b) = (0, 1/2)$ corresponds to temporal insertion of $e^{i\pi J_0^3}$ within the trace and by direct calculations we may show that it is related to the twisted $\mathcal{N} = 2$ characters by,¹³

$$\chi_{\ell,(0,1/2)}^{\text{SU}(2)_k}(\tau, 0) = (-1)^{\ell/2} \chi_{\ell,[0,1]}^k(\tau) . \quad (\text{C.2})$$

(Both $\chi_{\ell,(0,1/2)}^{\text{SU}(2)_k}(\tau, 0)$ and $\chi_{\ell,[0,1]}^k(\tau)$ vanish when ℓ is odd, so the factor $(-1)^{\ell/2}$ entails no phase ambiguity.) Performing modular transformations, we further obtain

$$\begin{aligned} \chi_{\ell,(1/2,0)}^{\text{SU}(2)_k}(\tau, 0) &= \chi_{\ell,[1,0]}^k(\tau), \\ \chi_{\ell,(1/2,1/2)}^{\text{SU}(2)_k}(\tau, 0) &= e^{2\pi i \frac{k}{16}} e^{-\frac{i\pi}{2}\ell} \chi_{\ell,[1,1]}^k(\tau) . \end{aligned} \quad (\text{C.3})$$

The modular property of the twisted character $\chi_{\ell,(a,b)}^{\text{SU}(2)_k}(\tau, z)$ is simply written as

$$\chi_{\ell,(a,b)}^{\text{SU}(2)_k}(-1/\tau, z/\tau) = e^{i\pi \frac{k}{2} \frac{z^2}{\tau}} \sum_{\ell'=0}^k S_{\ell,\ell'} \chi_{\ell',(b,-a)}^{\text{SU}(2)_k}(\tau, z), \quad (\text{C.4})$$

$$\chi_{\ell,(a,b)}^{\text{SU}(2)_k}(\tau + 1, z) = e^{2\pi i \left(\frac{\ell(\ell+2)}{4(k+2)} - \frac{k}{8(k+2)} \right)} \chi_{\ell,(a,a+b)}^{\text{SU}(2)_k}(\tau, z) . \quad (\text{C.5})$$

D. Complete classification of twisted $\mathcal{N} = 4$ characters

In this appendix we present a complete classification of the twisted $\mathcal{N} = 4$ characters.

- **$\sigma_1^{\mathcal{N}=4}$ -twist.** As we already discussed, a major part of the $\sigma_1^{\mathcal{N}=4}$ -twisted characters are exhibited in (2.29). We thus focus on the remaining sectors. It is enough to consider the $\{\mathbb{R}, [0, 1]\}$ ($\{\tilde{\mathbb{R}}, [0, 1]\}$) sector, which is the trace over each Ramond

¹³Since $\chi_{\ell,(a/2,b/2)}^{\text{SU}(2)_k}(\tau, 0)$ and $\chi_{\ell,[[a],[b]]}^k(\tau)$ ($[a] \in \mathbb{Z}_2$ is defined by $a \equiv [a] \pmod{2}$) differ only by a phase factor, $\chi_{\ell,[[a],[b]]}^k(\tau)$ may also be regarded as twisted $SU(2)_k$ characters. In fact, the same character functions $\chi_{\ell,[[a],[b]]}^k(\tau)$ are employed in [6] to analyse twisted representations that display manifest \mathbb{Z}_2 -periodicities in the twist parameters. In that paper, formulas involving the angular variable dependence (z) associated with the $SU(2)$ zero-modes are presented. In the $\mathcal{N} = 2$ case the z -dependence is irrelevant because the $\mathcal{N} = 2$ -involution $\sigma^{\mathcal{N}=2}$ removes the zero-mode of the $U(1)$ -current J .

representation with $\sigma_1^{\mathcal{N}=4} ((-1)^F \sigma_1^{\mathcal{N}=4})$ inserted. The remaining ones $\{\widetilde{\text{NS}}, [1, 0]\}$, $\{\text{NS}, [1, 1]\}$ ($\{\widetilde{\text{R}}, [1, 0]\}$, $\{\widetilde{\text{R}}, [1, 1]\}$) are generated by modular transformations. As already mentioned, $\sigma_1^{\mathcal{N}=4}$ boils down to the $\sigma^{\mathcal{N}=2}$ -twist and the spectral-flowed sectors do not contribute. Thus we find

$$\text{Tr}_{\mathcal{C}_h^{(\text{R})}} \left[\sigma_1^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] = \text{Tr}_{\mathcal{D}_{1/2}^{(\text{R})}} \left[\sigma_1^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] = 0, \quad \text{Tr}_{\mathcal{D}_0^{(\text{R})}} \left[\sigma_1^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] = \pm 1. \quad (\text{D.1})$$

We also obtain the same results for the $\{\widetilde{\text{R}}, [0, 1]\}$ -characters. It is trivial to modular transform these results to obtain the remaining ones.

- **$\sigma_3^{\mathcal{N}=4}$ -twist.** The equivalence of twisted character formulas for $\sigma_3^{\mathcal{N}=4}$ and $\sigma_1^{\mathcal{N}=4}$ is anticipated; we shall verify this explicitly.

Again we focus on the $\{\text{R}, [0, 1]\}$ and $\{\widetilde{\text{R}}, [0, 1]\}$ sectors, since the classification (2.29) has been already given. Namely, we examine the trace over each Ramond representation with the insertion of $\sigma_3^{\mathcal{N}=4}$ (and $(-1)^F \sigma_3^{\mathcal{N}=4}$), which assigns the phase $(-1)^n$ to the n -th spectral flow sector. For representations $\mathcal{C}_h^{(\text{R})}$, $\mathcal{D}_{1/2}^{(\text{R})}$, ($\mathcal{C}_h^{(\widetilde{\text{R})}$, $\mathcal{D}_{1/2}^{(\widetilde{\text{R})})}$) we readily obtain

$$\text{Tr}_{\mathcal{C}_h^{(\text{R})}} \left[\sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] \left(\equiv \text{Tr}_{\mathcal{C}_h^{(\text{R})}} \left[(-1)^F \sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] \right) = q^{h - \frac{3}{8}} \frac{i\theta_1(\tau, 0)\theta_2(\tau, 0)}{\eta(\tau)^3} = 0, \quad (\text{D.2})$$

$$\text{Tr}_{\mathcal{D}_{1/2}^{(\text{R})}} \left[\sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] \left(\equiv \text{Tr}_{\mathcal{D}_{1/2}^{(\text{R})}} \left[(-1)^F \sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] \right) = q^{-\frac{1}{8}} \frac{i\theta_1(\tau, 0)\theta_2(\tau, 0)}{\eta(\tau)^3} = 0, \quad (\text{D.3})$$

by using (2.22), (2.24) and the 1/2-spectral flow. The one for the representation $\mathcal{D}_0^{(\text{R})}$ is somewhat non-trivial:

$$\text{Tr}_{\mathcal{D}_0^{(\text{R})}} \left[\sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] = \pm \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} \theta_2(\tau)}{1 + q^n} \frac{1}{\eta(\tau)^3} \equiv \pm 1. \quad (\text{D.4})$$

(Again we include a sign ambiguity due to the $\sigma_3^{\mathcal{N}=4}$ -action on the vacuum.) The second equality follows from the identity (e.g. (3.17) in [30])

$$\frac{1}{\prod_{n=1}^{\infty} (1 + yq^{n-\frac{1}{2}})(1 + y^{-1}q^{n-\frac{1}{2}})} = \frac{q^{\frac{1}{12}}}{\eta(\tau)^2} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{1 + yq^{n+\frac{1}{2}}}, \quad (\text{D.5})$$

which can be derived from the super boson-fermion correspondence [30, 31].¹⁴

On the other hand, if making the $(-1)^F \sigma_3^{\mathcal{N}=4}$ -insertion, only the Ramond ground states can contribute, and it is easy to see

$$\text{Tr}_{\mathcal{D}_0^{(\text{R})}} \left[(-1)^F \sigma_3^{\mathcal{N}=4} q^{L_0 - \frac{1}{4}} \right] = \pm 1. \quad (\text{D.6})$$

¹⁴More general identities for the level k Appell function $\mathcal{K}_k(\tau, \nu, \mu) \equiv \sum_{n \in \mathbb{Z}} \frac{q^{\frac{k}{2}n^2} x^{kn}}{1 - xyq^n}$, ($x \equiv e^{2\pi i\nu}$, $y \equiv e^{2\pi i\mu}$) are given in [32, 33]. The $k = 1$ case [32] is relevant here:

$$\theta_3(\tau, \lambda)\mathcal{K}_1(\tau, \nu, \mu) - \theta_3(\tau, \nu)\mathcal{K}_1(\tau, \lambda, \mu) = i \frac{\theta_3(\tau, \nu + \mu + \lambda)\theta_1(\tau, -\nu + \lambda)}{\theta_1(\tau, \nu + \mu)\theta_1(\tau, \mu + \lambda)} \eta(\tau)^3,$$

from which one can reproduce the identity (D.5) by setting $\lambda = \frac{\tau+1}{2}$. Generalization to higher level cases has been given in [33] (Lemma 2.2).

In this way we have confirmed the equality of the $\sigma_1^{\mathcal{N}=4}$ - and $\sigma_3^{\mathcal{N}=4}$ -twisted characters for all the irreducible representations of $\mathcal{N} = 4$ SCA.

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